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# Higher spin superalgebras in any dimension and their representations

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Abstract: Fock module realization for the unitary singleton representations of the $d-1$ dimensional conformal algebra $o(d-1,2)$, which correspond to the spaces of single-particle states of massless scalar and spinor in $d-1$ dimensions, is given. The pattern of the tensor product of a pair of singletons is analyzed in any dimension. It is shown that for $d > 3$ the tensor product of two boson singletons decomposes into a sum of all integer spin totally symmetric massless representations in $AdS_d$, the tensor product of boson and fermion singletons gives a sum of all half-integer spin symmetric massless representations in $AdS_d$, and the tensor product of two fermion singletons in $d > 4$ gives rise to massless fields of mixed symmetry types in $AdS_d$ depicted by Young tableaux with one row and one column together with certain totally antisymmetric massive fields. In the special case of $o(2,2)$, tensor products of $2d$ massless scalar and/or spinor modules contain infinite sets of $2d$ massless conformal fields of different spins. The obtained results extend the $4d$ result of Flato and Fronsdal [1] to any dimension and provide a nontrivial consistency check for the recently proposed higher spin model in $AdS_d$ [2]. We define a class of higher spin superalgebras which act on the supersingleton and higher spin states in any dimension. For the cases of $AdS_3$, $AdS_4$, and $AdS_5$ the isomorphisms with the higher spin superalgebras defined earlier in terms of spinor generating elements are established.

Keywords: Field Theories in Higher Dimensions, AdS-CFT and dS-CFT Correspondence.



 

## Contents



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## 1. Introduction

In the paper [2] nonlinear equations of motion for interacting totally symmetric massless bosonic fields of all spins in any dimension have been formulated. The primary goal of this paper is to show that the global higher spin (HS) symmetry algebras of [2] admit massless unitary representations which correspond to the sets of massless fields of the models of [2]. This provides a nontrivial consistency check of the results of [2] analogous to that carried out in [3, 4] for the $4d$ HS models.

One of the key results of this paper consists of the extension to any dimension of the theorem of Flato and Fronsdal [1] which states that the tensor products of pairs of $AdS_4$ singletons give rise to sums of all $AdS_4$ massless representations of $o(3,2) \sim sp(4)$. The generalization of the Flato-Fronsdal theorem for the $AdS_5$ case, which is of most interest from the superstring theory side, was obtained in [5] in terms of doubletons (see also [6]). Analogous analysis of the case of $AdS_7$ was performed in [7]. The key element for these constructions was the oscillator realization of the space-time symmetry algebras and their superextensions based on the low-dimensional isomorphisms like $o(2,2) \sim sp(2) \oplus sp(2)$, $o(3,2) \sim sp(4)$ and $o(4,2) \sim hu(1,1|(0,1,2):[M,2])(2,2)$ which allow realizations of space-time superalgebras in terms of bilinears of oscillators carrying spinor representations of space-time symmetry (super)algebras. The singleton and doubleton representations in lower dimensions admit a simple realization of Fock modules associated with the these spinor oscillators (see [8] and references therein). In these terms the Flato-Fronsdal theorem can be proved directly by decomposing the tensor product of two such Fock modules into irreducible submodules of the same symmetry algebra (see for example [4] for the $AdS_4$ case). However, the realization in terms of spinors does not work beyond some lower dimensions because the afore mentioned isomorphisms do not take place for general $d$. Nevertheless, as we show, the analysis can be performed in any dimension within the realization of the orthogonal algebra $o(M,2)$ in terms of bosonic oscillators carrying $o(M,2)$ vector indices. In this case the corresponding Fock module also plays the key role. The difference between the two constructions is that this Fock module forms a reducible $o(M,2)$-module and, to single out the unitary singleton submodules, some additional restrictions on the carrier space have to be imposed. A somewhat unusual feature is that the corresponding submodules do not contain the Fock vacuum. Otherwise, the extension of the Flato-Fronsdal theorem to any dimension is quite uniform.

The Flato-Fronsdal theorem and its higher dimensional extensions provide a group-theoretical basis for the $AdS/CFT$ correspondence conjecture [9, 10, 11] and are especially important for the analysis of the correspondence between $d$ dimensional boundary conformal models and HS gauge models in the bulk $AdS_{d+1}$. The latter issue was addressed in a number of papers in different contexts [12]–[27]. (A closely related issue is the analysis of the tensionless limit of string in AdS; see, for example, [28]–[31].) Our group-theoretical analysis agrees with the results of [23, 26], where the sector of a boundary conformal scalar field in any dimension was discussed, and suggests the extension of these results to the models with boundary and bulk fermions in any dimension.

The bosonic HS algebra of [2] is the conformal HS algebra of a massless boundary scalar [32] in $d-1$ dimensions, i.e. it is the infinite dimensional symmetry algebra of the massless Klein-Gordon equation. Another class of HS algebras is associated with the massless boundary spinors. In the case of $d=4$ these two algebras were isomorphic. As we show this is not true beyond $d=4$. The $AdS_d$ bulk gauge fields corresponding to bilinears of massless boundary scalar form the set of all totally symmetric massless bosonic fields. The $AdS_d$ bulk gauge fields of the conformal HS algebras corresponding to bilinears of a boundary spinor are bosons having mixed symmetry described by Young tableaux with one row and one column of various lengths and heights, respectively. For odd $d$ (i.e., even-dimensional boundary theory) there are generically three sorts of HS algebras: nonchiral type $A$ algebras and two type $B$ chiral algebras which correspond to chiral boundary spinors. In the type $B$ cases antisymmetric HS bulk tensors satisfy certain (anti)selfduality conditions.

Not surprisingly, the scalar×scalar and spinor×spinor HS algebras are two bosonic subalgebras of some HS superalgebra in $AdS_d$ with any $d$. The fermionic sector of the corresponding bulk $AdS_d$ gauge fields consists of all totally symmetric half-integer spin



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fields in $AdS_d$. We will argue that all constructed (super)algebras underly some consistent HS gauge theories in $AdS_d$. It is important to note that the infinite dimensional HS superalgebras constructed in this paper contain finite dimensional SUSY subalgebras only for some lower dimensions that admit equivalent description in terms of spinor twistor variables.

The content of this paper is as follows. In the rest of the Introduction we summarize some relevant facts on the unitary representations of the $AdS_d$ algebra $o(d-1,2)$ (subsection 1.1) and discuss some general properties of the HS algebras (subsection 1.2) focusing main attention on the admissibility condition which gives a criterion that allows one to single out those algebras which can be symmetries of a consistent field-theoretical model. In section 2 we define simplest bosonic HS algebras. The projection technics useful for the analysis of quotient algebras is introduced in section 3. The HS superalgebras are defined in section 4. In section 5 it is shown that for the particular case of $AdS_4$ the HS superalgebra admits equivalent realization of [53, 54] in terms of spinors, and analogous construction is discussed for the $AdS_3$ and $AdS_5$ HS algebras. Oscillator (Fock) realization for the unitary representation of single-particle states of the boundary conformal scalar and spinor are constructed in sections 6 and 7, respectively. In section 8, the pattern of the tensor products of these modules is found and it is shown that HS superalgebras discussed in this paper satisfy the admissibility condition. Unfolded formulation of the free field equations for boundary conformal fields is briefly discussed in section 9. Section 10 contains conclusions. In appendix we collect some useful facts on the description of Young tableaux in terms of oscillators.

### 1.1 Anti-de Sitter algebra

HS algebras are specific infinite dimensional extensions of one or another $d$ dimensional space-time symmetry (super)algebra $g$. In this paper we will be mainly interested in the $AdS_d$ case of $g = o(d-1,2)$. The generators $T^{AB}$ of $o(M,2)$ satisfy the commutation relations

$$[T^{AB}, T^{CD}] = \eta^{BC}T^{AD} - \eta^{AC}T^{BD} - \eta^{BD}T^{AC} + \eta^{AD}T^{BD}\,, \tag{1.1}$$

where $\eta^{AB}$ is the invariant symmetric form of $o(M,2)$ ($A, B = 0, \ldots, M+1$). We will use the mostly minus convention with $\eta^{00} = \eta^{M+1M+1} = 1$ and $\eta^{ab} = -\delta^{ab}$ for the space-like values of $A = a = 1 \ldots M$. The $AdS_{M+1}$ energy operator is

$$E = iT^{M+1\,0}\,. \tag{1.2}$$

The noncompact generators of $o(M,2)$ are

$$T^{\pm a} = iT^{0a} \mp T^{M+1\,a}\,, \tag{1.3}$$

$$[E, T^{\pm a}] = \pm T^{\pm a}\,, \qquad [T^{-a}, T^{+b}] = 2(\delta^{ab}E + T^{ab})\,. \tag{1.4}$$

The compact generators $T^{ab}$ of $o(M)$ commute with $E$. The generators $T^{AB}$ are antihermitean, $(T^{AB})^\dagger = -T^{AB}$, and, therefore,

$$E^\dagger = E\,, \qquad (T^{\pm a})^\dagger = T^{\mp a}\,, \qquad (T^{ab})^\dagger = -T^{ab}\,. \tag{1.5}$$

An irreducible bounded energy unitary representation $\mathcal{H}(E_0, \mathbf{h})$ of $o(M,2)$ is characterized by some eigenvalue $E_0$ of $E$ and weight $\mathbf{h}$ of $o(M)$ which refer to the lowest energy (vacuum) states $|E_0, \mathbf{h}\rangle$ of $\mathcal{H}(E_0, \mathbf{h})$ that satisfy $T^{-a}|E_0, \mathbf{h}\rangle = 0$ and form a finite dimensional module of $o(M) \oplus o(2) \subset o(M,2)$. A value of the quadratic Casimir operator $C_2 = -\frac{1}{2}T^{AB}T_{AB}$ on $\mathcal{H}(E_0, \mathbf{h})$ is

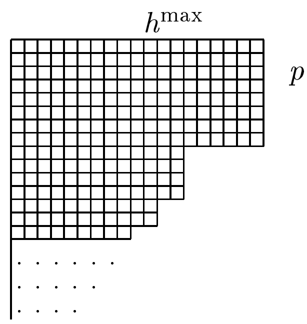
$$C_2 = E_0(E_0 - M) + \gamma(\mathbf{h}) \,, \tag{1.6}$$

where $\gamma(\mathbf{h})$ is the value of the Casimir operator $\gamma_2 = -\frac{1}{2}T^{ab}T_{ab}$ of $o(M)$ on the vacuum $|E_0, \mathbf{h}\rangle$.

As shown by Metsaev [33], a bosonic massless field in $AdS_{M+1}$, which carries "spin" corresponding to the representation of $o(M)$ with the weights $\mathbf{h}$, has the vacuum energy

$$E_0^{bos}(\mathbf{h}) = h^{\max} - p - 1 + M \tag{1.7}$$

where $h^{\max}$ is the length of the first row of the $o(M)$ Young tableau associated with the vacuum space $|E_0, \mathbf{h}\rangle$ while $p$ is the number of rows of length $h^{\max}$ at the condition that the total number of rows (i.e., $o(M)$ weights) does not exceed $\frac{1}{2}M$ (that can always be achieved by dualization with the help of the epsilon symbol). In other words $h^{\max}$ and $p$ are, respectively, the length and height of the upper rectangular block of the $o(M)$ Young tableau associated with the vacuum weight $\mathbf{h}$ (relevant definitions and facts on Young tableaux are collected in the appendix),




(1.8)

The expression for lowest energies of fermionic massless representations is analogous [34]

$$E_0^{fer}(\mathbf{h}) = h^{\max} - p - \frac{3}{2} + M \tag{1.9}$$

where, again, $h^{\max}$ and $p$ are, respectively, the length and height of the upper rectangular block of the $o(M)$ Young tableau associated with the $\gamma$-transverse tensor-spinor realization of the vacuum space (i.e., $|E_0, \mathbf{h}\rangle$ is realized as a space of $o(M)$ tensors carrying an additional $o(M)$ spinor index, with the $o(M)$ invariant tracelessness, $\gamma$-transversality and Young antisymmetry conditions imposed). A total number of rows of the corresponding Young tableau does not exceed $\frac{1}{2}M$.

More generally, let $D(E_0, \mathbf{h})$ be a generalized Verma module induced from some irreducible $o(M) \oplus o(2)$ vacuum module $|E_0, \mathbf{h}\rangle$. It is spanned by the states

$$T^{+a_1} \dots T^{+a_n}|E_0, \mathbf{h}\rangle \tag{1.10}$$

with various levels $n$. For the unitary case $D(E_0, \mathbf{h}) = \mathcal{H}(E_0, \mathbf{h})$ it is isomorphic to the Hilbert space of single-particle states of one or another field-theoretical system, i.e, a space of normalizable positive-energy solutions of some (irreducible) $o(M, 2)$ invariant field equations in $M + 1$ dimensional space-time.[1] Unitarity implies existence of some invariant positive-definite norm with respect to which the Hermiticity conditions (1.5) are satisfied. This requires the vacuum energy $E_0$ to be high enough

$$E_0 \geq E_0(\mathbf{h}) \,, \tag{1.11}$$

where $E_0(\mathbf{h})$ is some weight dependent minimal value of $E_0$ compatible with unitarity. Note that from the second relation in (1.4) it follows that $E_0(\mathbf{h}) \geq 0$ in a unitary module.

Starting from inside of the unitarity region and decreasing $E_0$ for a fixed $\mathbf{h}$ one approaches the boundary of the unitarity region, $E_0 = E_0(\mathbf{h})$. Some zero-norm vectors then appear in $D(E_0(\mathbf{h}), \mathbf{h})$ for $E_0 = E_0(\mathbf{h})$. These necessarily should have vanishing scalar product with any other state (otherwise there will be a negative norm state in contradiction with the assumption that $E_0$ is at the boundary of the unitarity region). Therefore, the zero-norm states form an invariant subspace called singular submodule $\mathcal{S}$. By factoring out this subspace one is left with some unitary module $\mathcal{H}(E_0(\mathbf{h}), \mathbf{h}) = D(E_0(\mathbf{h}), \mathbf{h})/\mathcal{S}$. Note that the submodule $\mathcal{S}$ is induced from some singular vectors $|E_0', \mathbf{h}'\rangle \in D(E_0(\mathbf{h}), \mathbf{h})$ among the states (1.10) which themselves satisfy the vacuum condition $T^{-a}|E_0', \mathbf{h}'\rangle = 0$.

It is well known that the appearance of the null subspace $\mathcal{S}$ manifests gauge symmetries in the underlying field-theoretical model. More precisely, $\mathcal{S}$ represents leftover on-mass-shell symmetries with the gauge parameters analogous to the leftover gauge symmetries of the Maxwell theory $\delta A_n = \partial_n \phi$ in the Lorentz gauge $\partial_n A^n = 0 \to \Box \phi = 0$. This is because the space (1.10) has some fixed values of all Casimir operators determined by the weights of the vacuum state $|E_0, \mathbf{h}\rangle$. As a result, the submodule $\mathcal{S}$ must have the same values of the Casimir operators. This means in particular that the states of $\mathcal{S}$ satisfy an appropriate Klein-Gordon equation associated with the quadratic Casimir of $o(M, 2)$. As pointed out by Flato and Fronsdal [36], gauge symmetries related with the singular modules can be of two different types.

Type I is the case of usual gauge symmetry allowing to gauge away some part of the $AdS_{M+1}$ bulk degrees of freedom of a field associated with the module $D(E_0(\mathbf{h}), \mathbf{h})$ so that the quotient module $\mathcal{H}(E_0(\mathbf{h}), \mathbf{h}) = D(E_0(\mathbf{h}), \mathbf{h})/\mathcal{S}$ describes a field with local degrees of freedom in $AdS_{M+1}$. The corresponding fields are gauge fields in $AdS_{M+1}$. We will call them massless fields as they have minimal lowest energies compatible with unitarity. Note that the relations (1.7), (1.9) for lowest energies of massless fields were derived by Metsaev [33, 34] just from the requirements of on-mass-shell gauge invariance of the corresponding massless equations along with the unitarity condition.[2] (For more details on the structure of unitary representations of noncompact algebras we refer the

[1] There are as many independent states (1.10) as on-mass-shell independent derivatives of any order of the dynamical fields under consideration at some point of a $M$ dimensional Cauchy surface.

[2] Let us note that partially massless fields in $AdS_d$ considered in [35] correspond to nonunitary $o(M, 2)$-modules resulting from factorization of submodules of pure gauge states in the appropriate generalized Verma modules.

reader to [37] and references therein). The massless representations of $o(M,2)$ of this class are those with $p < \frac{M}{2}$, i.e. the corresponding vacuum spaces are described by any $o(M)$-module except for those described by the rectangular Young tableaux of the maximal height $\frac{1}{2}M$ and an arbitrary length. Note that the latter representations exist only for even $M$ except for the degenerate cases of tableaux of zero length which correspond to the lowest energy scalar and spinor $o(M)$-modules for any $M$.

Type II is the case of boundary conformal fields which we will call singletons when discussing the corresponding unitary representations. This is the case where all bulk degrees of freedom are factored out so that the module $\mathcal{H}(E_0(\mathbf{h}),\mathbf{h})$ describes a dynamical system at the boundary of $AdS_{M+1}$. In this case, $o(M,2)$ acts as conformal group in $M$ dimensions. In accordance with the results of [38, 39] (see also [40]), the type-II representations are those with $p = \frac{M}{2}$, $M$ even, and minimal energy scalar and spinor $o(M)$ modules, i.e. the corresponding Young tableau is some rectangular of the maximal height $\frac{M}{2}$ and an arbitrary length (including zero). Indeed, it is easy to see that these fields form massless representations of $o(M-1,2)$: dualization of a height $\frac{1}{2}M$ tableau with respect to $o(M-1)$ gives a rectangular block of height $\frac{1}{2}M-1$ that just compensates the effect of replacing $M$ by $M-1$ in (1.7) and (1.9). Also the appearance of gauge degrees of freedom in the scalar or spinor modules indicates decoupling of bulk degrees of freedom [36]. Note that field-theoretical realization of this phenomenon was originally discovered by Dirac [43] for the case of $o(3,2)$.

Finally, let us make the following remark. Every lowest weight unitary $o(M,2)$-module spanned by the vectors (1.10) forms a unitary module of $o(M-1,2) \subset o(M,2)$. To find out its $o(M-1,2)$ pattern one has to decompose a $o(M,2)$-module $D(E_0,\mathbf{h})$ into a direct sum of $o(M-1,2)$-modules. This can be achieved by looking for vacuum states among (1.10) as those satisfying $T^{-a'}|E_0',\mathbf{h}'\rangle = 0$ with $a' = 1\dots M-1$. In the $o(M-1)$ covariant basis the states (1.10) are equivalent to

$$T^{+a'_1}\dots T^{+a'_n}(t^+)^m|E_0,\mathbf{h}\rangle\,, \qquad t^+ = T^{+M}\,. \tag{1.12}$$

Clearly, the dependence on $t^+$ results in the infinite reducibility of $D(E_0,\mathbf{h})$ treated as $o(M-1,2)$-module. This is expected because infinite towers of Kaluza-Klein modes should appear. On the other hand, this tower may be treated as an infinite dimensional module of the $M$ dimensional conformal algebra $o(M,2)$ which mixes fields of different nonzero masses in $AdS_M$.

### 1.2 General conditions on higher spin algebras

HS algebras are specific infinite dimensional extensions of one or another $d$ dimensional space-time symmetry (super)algebra $g$ which in the $AdS_d$ case is $o(d-1,2)$. Not every extension $h$ of $g$ gives a HS algebra, however. Here we summarize some general conditions to be satisfied by HS algebras which may help to rule out some candidates.

Generally speaking, a HS algebra is any unbroken global symmetry algebra $h$ of a vacuum solution of some consistent interacting theory which contains massless higher spins in the symmetric vacuum under consideration. To make it possible to interpret the model in



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terms of relativistic fields carrying some masses and spins, the vacuum solution is demanded to be invariant under the conventional space-time symmetry $g \subset h$ (e.g, $g = o(M,2)$). As a global symmetry of a consistent model, $h$ must possess a unitary representation which contains all massless states in the model. The same time, any symmetry parameter of $h$ should me associated with some gauge field of a particular spin, which, upon quantization, gives rise to the Hilbert space of massless states of a given type. This leads to the nontrivial matching condition on $h$ called in [3] admissibility condition.

As an illustration let us recall the standard argument used to classify pure supergravity models. The first step is to say that supergravity results from gauging the supersymmetry algebra with the generators of (Minkowski or AdS) translations $P^a$, Lorentz transformations $L^{ab} = -L^{ba}$, supertransformations $Q^i_\mu$ ($i = 1 \dots \mathcal{N}$, index $\mu$ is spinorial) and global symmetries $T^{ij} = -T^{ji}$. One concludes that any SUGRA model has to describe a set of gauge fields which contains spin 2 massless field (graviton) described by the frame 1-form $h^a$ and Lorentz connection 1-form $\omega^{ab} = -\omega^{ba}$ identified with the gauge fields corresponding to $P^a$ and $L^{ab}$, spin 3/2 massless fields (gravitino) described by 1-form spinors $\psi^i_\mu$ which are gauge fields for $Q^i_\mu$, and spin 1 massless gauge fields $A^{ij}$ which correspond to $T^{ij}$.

The second step is to check whether the supersymmetry algebra admits a unitary representation with exactly these sets of massless states plus, may be, some lower spin states which are not described by gauge fields. The answer is well known (see e.g. [44]). For $4d$ SUGRA, for example, the result is that for $\mathcal{N} \leq 8$ there is a massless supermultiplet with the highest spin 2 and the required pattern of spins 3/2 and 1. (In fact, with the exception of $\mathcal{N} = 7$ when the corresponding supermultiplet turns out to be a $\mathcal{N} = 8$ supermultiplet). For $\mathcal{N} > 8$ (i.e., with more than total 32 supercharges) one finds that there is always a state of spin $s > 2$ in every massless supermultiplet. Since any such a field is a gauge field, this does not match the list of gauge fields of the usual SUSY algebra.

As long as it was not known that nontrivial theories which contain massless HS fields require a curved background with nonzero cosmological constant in a most symmetric vacuum [45], consistent theories of HS massless fields were not believed to exist and the appearance of HS fields in SUSY supermultiplets used to "rule out" supergravities with $\mathcal{N} > 8$. A more constructive alternative was to study if there exist some extensions of the usual SUSY algebras such that a set of corresponding gauge fields would match some of their massless unitary representations. The analysis along these lines opens a way towards infinite dimensional HS algebras and nontrivial HS gauge theories. It was originally applied in [3] to the $d = 4$ case. In [4] a full list of admissible $4d$ HS algebras was obtained. The corresponding nonlinear HS theories were constructed at the level of classical field equations in [46]. The aim of this paper is to extend this analysis to any $d$.

Let us now discuss the general case. Let an action $S(q)$ depend on some variables $q^\Omega(x)$ and be invariant under gauge transformations

$$\delta q^\Omega = r^\Omega(q;\varepsilon)\,, \qquad \delta S = 0\,, \tag{1.13}$$

where $\varepsilon^i(x)$ are infinitesimal gauge parameters. The gauge transformation $r^\Omega(q;\varepsilon)$ contains the variables $q^\Omega(x)$ and parameters $\varepsilon^i(x)$ along with their derivatives.



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Let $q_0^\Omega$ be some solution of the field equations

$$\frac{\delta S}{\delta q^\Omega}\Big|_{q=q_0} = 0\,. \tag{1.14}$$

Perturbative analysis assumes that $q^\Omega$ fluctuates near $q_0^\Omega$, i.e.

$$q^\Omega = q_0^\Omega + \eta q_1^\Omega\,, \tag{1.15}$$

where $\eta$ is some small expansion parameter. A vacuum solution $q_0^\Omega$ is not invariant under the gauge transformation (1.13). One can however address the question if there are some nonzero symmetry parameters $\varepsilon^i_{gl}(x)$ which leave the vacuum solution invariant. Usually, if such a leftover symmetry exists at all, the $x$-dependence of $\varepsilon^i_{gl}(x)$ turns out to be fixed in terms of values of $\varepsilon^i_{gl}(x_0)$ at any given point $x_0$. This is why the leftover symmetries are global symmetries. The vacuum is called symmetric if a number of its global symmetries is not smaller than the number of local symmetries in the model, i.e. $\varepsilon^i_{gl}(x_0)$ for some fixed $x_0$ are modules of a global symmetry of the vacuum.

By definition of the global symmetry transformation, $\delta_{gl} q_0^\Omega = r^\Omega(q_0,\varepsilon_0) = 0$. As a result

$$\delta^{gl} q_1^\Omega = \delta_0^{gl} q_1^\Omega + o(\eta)\,, \qquad \delta_0^{gl} q_1^\Omega = \frac{\delta r^\Omega(q,\varepsilon_0,\ldots)}{\delta q^\Lambda}\Big|_{q=q_0} q_1^\Lambda\,, \tag{1.16}$$

where $o(\eta)$ denotes terms of higher orders in $q_1$. The $\eta$-independent part of the transformation (1.16) acts linearly on the fluctuations $q_1^\Omega$ in a way independent of a particular form of the nonlinear part of the transformations (which is important for the analysis of interactions, however). Starting with some closed set of nonlinear gauge transformations,[3] the global symmetry forms some (may be open) algebra as well. Expanding the full action as

$$S = S_0(q_0) + \eta^2 S_2(q_0, q_1) + o(\eta^2) \tag{1.17}$$

(as usual, the term linear in $\eta$ is absent because $q_0$ solves the field equations (1.14)) one observes that the global symmetry $\delta_0^{gl} q_1^\Omega$ is a symmetry of the free action $S_2$ bilinear in the fields $q_1$. Once the full theory is consistent, single-particle quantum states of the theory span a Hilbert space, which forms a bounded energy unitary $h$-module. It should decompose into a sum of unitary $g$-modules of the space-time symmetry $g \subset h$. The decomposition into UIRREPs of $g$ gives a list of particles described by the free action $S_2$ as well as their quantum numbers like spins and masses.

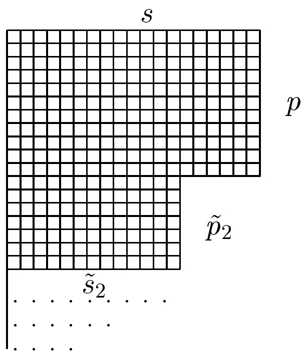
A particularly useful way to describe models with local symmetries is by using $p$-form gauge fields

$$\omega^i(x) = dx^{\underline{n}_1} \wedge \ldots \wedge dx^{\underline{n}_p} \omega^i_{\underline{n}_1\ldots\underline{n}_p}(x)\,. \tag{1.18}$$

In the recent paper [41] it was shown how this approach can be used to describe any $AdS_d$ massless representation associated with a gauge field. The rule is very simple. A massless module $\mathcal{H}(E_0(\mathbf{h}),\mathbf{h})$ with the $o(d-1)$ weight $\mathbf{h}$ corresponding to a finite dimensional

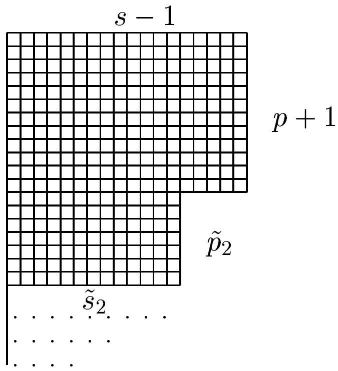
[3]These algebras may form so called open algebras with the commutators containing additional "trivial" transformations proportional to the left hand sides of the field equations.

$o(d-1)$-module formed by $o(d-1)$ traceless tensors with the symmetry properties of a Young tableau




(1.19)

can be described in terms of a $p$-form gauge field that takes values in the finite dimensional $o(d-1,2)$-module depicted by the $o(d-1,2)$ traceless Young tableau




(1.20)

which is obtained from (1.19) by cutting the right (i.e., shortest) column and adding the longest row. For example, to describe the spin two massless gauge field which corresponds to the $o(d-1)$ Young tableau $\square\!\square$, one introduces 1-form gauge field taking values in the representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of $o(d-1,2)$ to be interpreted as the gauge connection of $o(d-1,2)$. Its decomposition into representations of the Lorentz algebra $o(d-1,1)$ gives rise to the frame 1-form and Lorentz connection.

The sector of 1-forms is particularly important because the 1-form gauge fields take values in some (super)algebra $h$ with structure coefficients $f^i_{jk}$. The field strength is

$$R^i = d\omega^i(x) + f^i_{jk}\omega^j(x)\wedge\omega^k(x)\,, \qquad d = dx^{\underline{n}}\frac{\partial}{\partial x^{\underline{n}}}\,. \tag{1.21}$$

As just mentioned, gravitation is described in such a formalism by the components of the connection 1-form $\omega_0$ of the $AdS_d$ algebra $o(d-1,2)$. For background geometry with nondegenerate metric, $\omega_0(x)$ must be nonzero because it contains a nondegenerate frame field as the component of the gauge field $\omega^i(x)$ associated with the generator of space-time translations in $g$ (note that there exists invariant way to impose this condition with the help of the so called compensator formalism [47, 48, 49]). Gauge invariant lagrangians for gauge fields of any symmetry type can be built in terms of the field strengths of $\omega$ analogously to the MacDowell-Mansouri formulation for gravity [42].

Such a formalism has several nice properties. In particular, it allows a natural zero-curvature vacuum solution $\omega_0$ of the dynamical field equations such that

$$R^i(\omega_0) = 0\,. \tag{1.22}$$

For the case of gravity with the connection $\omega_0$ taking values in $o(d-1,2)$, this is just the equation for the $AdS_d$ space-time.

Any vacuum solution (1.22) has $h$ as a global symmetry (we disregard here possible global topological obstructions in less symmetric locally isomorphic spaces, extending, if necessary, the problem to the universal covering space-time). Indeed, the equation (1.22) is invariant under the gauge transformations

$$\delta\omega_0^i(x) = D_0\varepsilon^i(x) \equiv d\varepsilon^i(x) + f^i_{jk}\omega_0^j(x)\wedge\varepsilon^k(x)\,. \tag{1.23}$$

To have a fixed vacuum solution $\omega_0^i(x)$ invariant one requires

$$\delta\omega_0 = D_0(\varepsilon^i(x)) \equiv d\varepsilon^i(x) + f^i_{jk}\omega_0^j(x)\wedge\varepsilon^k(x) = 0\,. \tag{1.24}$$

The equation (1.24) is formally consistent because $D_0^2 = 0$. As a result it determines all derivatives of the 0-form $\varepsilon^i(x)$ in terms of its values $\varepsilon^i(x_0)$ at any given point $x_0$ of space-time. The corresponding parameters $\varepsilon_0^i(x)$, which solve (1.24), are fixed in terms of $\varepsilon_0^i(x_0)\in h$. They describe global symmetry $h$ of the vacuum (1.22). Let us note that the same argument is true for any gauge transformations which differ from (1.23) by terms proportional to the field strengths and/or matter (non-gauge) fields which are all assumed to have zero VEVs. Also, let us note that $p$-gauge forms have $(p-1)$-form gauge parameters. However, as a consequence of the Poincaré lemma only 1-form gauge fields give rise to nontrivial global symmetries with 0-form parameters.

Given Lie superalgebra $h$ we can check using the results of [41] whether or not it is appropriate to describe some set of HS gauge fields and, if yes, to find out the spectrum of spins of this set. Note that the condition that the gauge fields of $h$ correspond to some set of massless fields is itself nontrivial, imposing rigid restrictions on $h$. In particular, according to [41], a result of the decomposition of $h$ into irreducible submodules of the space-time symmetry $o(d-1,2)$ has to contain only finite dimensional[4] representations of $o(d-1,2)$ of special types, namely those depicted by traceless Young tableaux of $o(d-1,2)$ that have two first rows of equal length.

Suppose now that there is a consistent nonlinear theory of massless gauge fields formulated in terms of connections of some algebra $h$ plus, may be, some number of fields described by 0-forms as well as by higher forms. Consistent interactions may deform a form of the transformation law (1.23) by some terms proportional to the curvatures $R$ and/or matter fields. In the vacuum with zero curvature and matter fields the deformation terms

[4]This guarantees that the gauge fields corresponding to every $g$-submodule in $h$ have a finite number of components. Note that usually HS models describe infinite set of fields, each having a finite number of components to describe one or another particle.

do not contribute to the global symmetry transformation law. As a result, if a consistent nonlinear theory exists, $h$ is the global symmetry algebra of such its vacuum solution. This automatically implies that the space-time symmetry $g \subset h$ is also the global symmetry of the model. It is one of the advantages of the formulation in terms of gauge connections that it makes global symmetries of the model manifest including the usual space-time symmetries. Note that this is achieved in a coordinate independent way because it is not necessary to know a particular form of the vacuum gauge connection $\omega_0$. Instead, it is enough to impose (1.22) along with the condition that the frame field is nondegenerate (see [49] for more details for the example of gravity).

Once $h$ is the global symmetry algebra of a hypothetical consistent HS theory it has to obey the admissibility condition [3] that there should be a unitary $h$-module which describes a list of quantum single-particle states corresponding to all HS gauge fields described in terms of the connections of $h$. If no such a module exists, there is no chance to find a nontrivial consistent (in particular, free of ghosts) theory that admits $h$ as a symmetry of its most symmetric vacuum. On the other hand, once some (super)algebra $h$ satisfying the admissibility condition is found, the pattern of the appropriate unitary module also contains the information on the matter fields and higher form HS gauge fields analogous to the matter and higher form fields in extended supergravity supermultiplets. As a result, one obtains a list of ($p = 0$) matter and ($p > 1$)-form fields to be introduced to make it possible to build a consistent theory.

We see that *a priori* not every extension $h$ of $g$ is a HS algebra. In particular, it is interesting to check whether the algebras considered in [50] satisfy the formulated criteria. The admissibility condition is the necessary condition for $h$ to underly some consistent HS theory. In practice, all examples of algebras $h$ known so far, which satisfy the admissibility condition, turned out to be vacuum symmetries of some consistent HS theories. Let us note that the admissibility condition applies to the symmetry algebras of on-mass-shell single-particle states. It therefore does not provide any criterion on the structure of possible further extensions of the HS algebras with the additional fields being pure gauge or auxiliary, i.e., carrying no degrees of freedom. Such algebras may have physical relevance as off-mass-shell algebras giving rise to some auxiliary field variables which are zero by virtue of dynamical field equations.



## 2. Simplest bosonic higher spin algebras

The HS algebra used in [2] is defined as follows. Consider bosonic oscillators $Y_i^A$ with $i = 1, 2$

$$(Y_i^A)^\dagger = Y_i^A \tag{2.1}$$

satisfying the commutation relations

$$[Y_k^A, Y_j^B] = 2iC_{kj}\eta^{AB}\,, \tag{2.2}$$

where $C_{ij} = -C_{ji}$, $C^{ij}C_{ik} = \delta^j_k$. The bilinear forms $\eta^{AB}$ and $C^{ij}$ are used to raise and lower indices in the usual manner: $A^A = \eta^{AB}A_B$, $a^i = C^{ij}a_j$, $a_i = a^jC_{ji}$.

The generators of $o(M,2)$ are

$$T^{AB} = -T^{BA} = \frac{1}{4i}\{Y^{jA}, Y_j^B\}\,. \tag{2.3}$$

Also we introduce the generators of $sp(2)$

$$t_{kl} = \frac{1}{4i}\eta_{AB}\{Y_k^A, Y_l^B\}\,. \tag{2.4}$$

The generators $T^{AB}$ and $t_{ij}$ commute to the oscillators $Y_j^A$ as follows

$$[T^{AB}, Y_j^C] = \eta^{BC}Y_j^A - \eta^{AC}Y_j^B\,, \qquad [t_{jk}, Y_n^A] = C_{kn}Y_j^A + C_{jn}Y_k^A\,, \tag{2.5}$$

i.e., $T^{AB}$ generate $o(M,2)$ rotations of indices $A, B, \ldots$, while $t_{ij}$ generate $sp(2)$ transformations of indices $i, j, \ldots$ Because indices in (2.3) and (2.4) are contracted with the $sp(2)$ and $o(M,2)$ invariant forms, respectively, one finds that

$$[T^{AB}, t_{ij}] = 0\,, \tag{2.6}$$

i.e., $o(M,2)$ and $sp(2)$ are Howe dual [51]. Note that the following identity is true as a consequence of (2.3) and (2.4)

$$C_2 \equiv -\frac{1}{2}T^{AB}T_{AB} = -\frac{1}{4}(M^2-4) - \frac{1}{2}t^{ij}t_{ij}\,. \tag{2.7}$$

Consider the associative Weyl algebra $A_{M+2}$ spanned by the elements

$$f(Y) = \sum_m f_{A_1\ldots A_m}^{i_1\ldots i_m} Y_{i_1}^{A_1}\ldots Y_{i_m}^{A_m}\,, \tag{2.8}$$

i.e. $A_{M+2}$ is the enveloping algebra of the commutation relations (2.2). Consider the subalgebra $S \subset A_{M+2}$ spanned by the $sp(2)$ singlets

$$f(Y) \in S\ : \qquad [t_{ij}, f(Y)] = 0\,. \tag{2.9}$$

These conditions admit a nontrivial solution with $f(Y)$ being an arbitrary function of the $o(M,2)$ generators (2.3). Let $hc(1|2{:}[M,2])$ be the Lie algebra resulting from $S$ with the commutator $[a,b]$ as the Lie product. (In this notation, $2:[M,2]$ refers to the dual pair $sp(2)\oplus o(M,2)$, $h$ abbreviates "higher", $c$ abbreviates "centralizer", and 1 refers to a number of Chan-Paton indices as explained in section 4.)

The algebras $S$ and $hc(1|2{:}[M,2])$ contain two-sided ideals $I$ spanned by the elements of the form

$$g = t_{ij}g^{ij}\,, \qquad g \in S \tag{2.10}$$

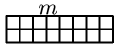
where $g^{ij}(Y)$ behaves as a symmetric tensor of $sp(2)$, i.e.,

$$[t_{ij}\,, g^{kl}] = \delta_j^k g_i{}^l + \delta_i^k g_j{}^l + \delta_j^l g_i{}^k + \delta_i^l g_j{}^k \tag{2.11}$$

(note that $t_{ij}g^{ji} = g^{ji}t_{ij}$). Actually, from (2.9) it follows that $fg$, $gf \in I$, $\forall f \in S$, $g \in I$. From the definition (2.4) of $t_{ij}$ one concludes that the ideal $I$ takes away all traces of the $o(M,2)$ tensors so that the algebra $S/I$ has only traceless $o(M,2)$ tensors in the expansion (2.8).

The $sp(2)$ invariance condition (2.9) is equivalent to

$$\Big(Y^{Ai}\frac{\partial}{Y^A_j}+Y^{Aj}\frac{\partial}{Y^A_i}\Big)f(Y)=0\,. \tag{2.12}$$

For the expansion (2.8), this condition implies that the coefficients $f^{\overbrace{1\ldots1}^{m}\overbrace{2\ldots2}^{n}}_{A_1\ldots A_m\,C_1\ldots C_n}$ are nonzero only if $n=m$ and that symmetrization over any $m+1$ indices among $A_1,\ldots A_m$, $C_1,\ldots C_m$ gives zero. This implies that the coefficients $f^{\overbrace{1\ldots1}^{m}\overbrace{2\ldots2}^{m}}_{A_1\ldots A_m\,C_1\ldots C_m}$ have the symmetry properties of the two-row rectangular Young tableau $m$  (for more details see appendix). Thus, the algebras $S$ and $hc(1|2{:}[M,2])$ decompose into direct sums of $o(M,2)$-modules described by various two-row rectangular Young tableaux. The algebra $S/I$ is spanned by the elements with traceless $o(M,2)$ tensor coefficients which have symmetry properties of two-row Young tableaux. The Lie algebra $hc(1|2{:}[M,2])/I$ was identified by Eastwood with the conformal HS algebra in $M$ dimensions in [32] where its realization in terms of the enveloping algebra of $o(M,2)$ was used. In [2] the algebra $hc(1|2{:}[M,2])/I$ was called $hu(1|2{:}[M,2])(1/sp(2)[M,2])$. To simplify notations we call this algebra $hu(1|2{:}[M,2])$ in this paper. More precisely, $hu(1|2{:}[M,2])$ is the real Lie algebra singled out by the reality condition

$$f(Y)\in hu(1|2{:}[M,2]):\qquad f(Y)\in hc(1|2{:}[M,2])/I\,,\quad f^\dagger(Y)=-f(Y)\,, \tag{2.13}$$

where $\dagger$ is the involution (2.1) extended to a generic element by the standard properties $(fg)^\dagger=g^\dagger f^\dagger$, $(\lambda g)^\dagger=\bar\lambda f^\dagger$ for $f,g\in hc(1|2{:}[M,2])/I,\lambda\in\mathbf{C}$. Note that $hu(1|2{:}[M,2])$ contains the $o(M,2)$ generators $T^{AB}$ (2.3).

Let us now consider another Lie algebra $hu(1|(1,2){:}[M,2])$ resulting from the analogous construction with the additional Clifford elements $\phi^A$ satisfying the anticommutation relations

$$\{\phi^A,\phi^B\}=-2\eta^{AB}\,. \tag{2.14}$$

Now the generators of $o(M,2)$ are realized as

$$T^{AB}=-T^{BA}=\frac{1}{4i}\{Y^{jA},Y_j{}^B\}-\frac{1}{4}[\phi^A,\phi^B]\,. \tag{2.15}$$

They commute to the (super)generators of $osp(1,2)$

$$t_j=\frac{1}{2}Y^A_j\phi_A\,,\qquad t_{nm}=i\{t_n,t_m\}=\frac{1}{4i}\eta_{AB}\{Y^A_n,Y^B_m\}\,. \tag{2.16}$$

The associative Weyl-Clifford algebra $A_{M+2,M+2}$ is spanned by the elements

$$f(Y,\phi)=\sum_{m,n,k}f_{A_1\ldots A_m\,,B_1\ldots B_n;C_1\ldots C_k}Y_1^{A_1}\cdots Y_1^{A_m}Y_2^{B_1}\cdots Y_2^{B_n}\phi^{C_1}\cdots\phi^{C_k}\,, \tag{2.17}$$

where the coefficients $f_{A_1\ldots A_m\,,B_1\ldots B_n;C_1\ldots C_k}$ are totally symmetric (separately) in the indices $A_1\ldots A_m$ and $B_1\ldots B_n$ and totally antisymmetric in the indices $C_1\ldots C_k$. Its subalgebra $\mathcal{S}$ is spanned by the elements which have zero graded commutator with the (super)generators of $osp(1,2)$, i.e.

$$f\in\mathcal{S}:\quad f(Y,\phi)t_j=t_jf(Y,-\phi)\,. \tag{2.18}$$

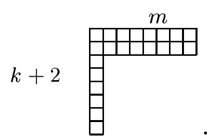
From (2.16) it follows that any $f\in\mathcal{S}$ is $sp(2)$ invariant. As a result, $n=m$ and $f_{A_1\ldots A_m\,,B_1\ldots B_m;C_1\ldots C_k}$ has the symmetry properties of a two-row rectangular Young tableau with respect to the indices $A_1\ldots A_m$ and $B_1\ldots B_m$, i.e. symmetrization over any $m+1$ indices $A_i$ and $B_j$ gives zero. The condition (2.18) implies

$$(k+1)f_{A_1\ldots A_m\,,B_1\ldots B_m;A_{m+1}C_1\ldots C_k}=i(m+1)f_{A_1\ldots A_{m+1}\,,B_1\ldots B_mC_1;C_2\ldots C_k}\,, \tag{2.19}$$

where total (anti)symmetrization over the indices $(C)A$ is understood. Its general solution is

$$\begin{aligned}f_{A_1\ldots A_m\,,B_1\ldots B_m;C_1\ldots C_k}&=g_{A_1\ldots A_m\,,B_1\ldots B_m|C_1\ldots C_k}\\&\quad+i\theta(k-2)\frac{(m+1)^2}{m+k}g_{A_1\ldots A_mC_1\,,B_1\ldots B_mC_2|C_3\ldots C_k}\,,\end{aligned} \tag{2.20}$$

where $g_{A_1\ldots A_m\,,B_1\ldots B_m|C_1\ldots C_k}$ is an arbitrary tensor that has symmetry properties of the Young tableau




. (2.21)

In other words $g_{A_1\ldots A_m\,,B_1\ldots B_m|C_1\ldots C_k}$ is totally symmetric in the indices $A_1\ldots A_m$ and $B_1\ldots B_m$, totally antisymmetric in the indices $C_1\ldots C_k$ and such that symmetrization over any $m+1$ indices gives zero.

Let $hc(1|(1,2)$:$[M,2])$ be the Lie algebra isomorphic to $\mathcal{S}$ as a linear space, with the commutator (not graded commutator!) in $\mathcal{S}$ as the Lie bracket. The factorization over the ideal $I$ spanned by the elements of $hc(1|(1,2)$:$[M,2])$ which are themselves proportional to the generators of $osp(1,2)$ implies that the tableaux (2.21) are traceless. The resulting algebra $hu(1|(1,2)$:$[M,2])$ decomposes as a linear space into the direct sum of traceless representations of $o(M,2)$ which have the symmetry properties of the Young tableaux (2.21). More precisely, $hu(1|(1,2)$:$[M,2])$ is the real Lie algebra spanned by the elements satisfying

$$(f(Y,\phi))^\dagger=-f(Y,\phi) \tag{2.22}$$

at the condition that (2.1) is true along with

$$(\phi_A)^\dagger=-\phi_A\,. \tag{2.23}$$

(Let us note that the involution $\dagger$ can be realized in the usual manner as $(a)^\dagger=\phi^0\phi^{M+1}a^+\times\phi^{M+1}\phi^0$, where $a^+$ is some hermitean conjugation with respect to a positive-definite form.

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Note that a sign on the right hand side of (2.23) is chosen so that the space-like components of $\phi_A$ could be realized by hermitean matrices).

From the formula

$$\epsilon_{n_1\dots n_{M+2}}\epsilon_{m_1\dots m_{M+2}} = \sum_p (-1)^{\pi(p)}\eta_{n_1 m_{p(1)}}\dots\eta_{n_{M+2}m_{p(M+2)}}\,, \tag{2.24}$$

where summation is over all permutations $p$ of indices $m_i$ and $\pi(p) = 0$ or 1 is the oddness of the permutation $p$, it follows that any traceless tensor with the symmetry properties of a Young tableau, which contains two columns with more that $M+2$ cells, is identically zero.[5] From here it follows that only Young tableaux with up to $M$ cells in the first column appear among the $o(M,2)$ representations contained in $hu(1|(1,2)\text{:}[M,2])$, having at least two nonzero columns. It turns out however that elements described by the one-column Young tableaux of height 1 and $M+1$ belong to the ideal $I$ and, therefore, do not appear among the elements of $hu(1|(1,2)\text{:}[M,2])$. This fact is the content of *Lemma 3.1* of section 3. Let us mention that the feature that factoring out elements proportional to $t_i$ may imply some factorization beyond only taking away traces is because Clifford algebra is finite dimensional. In other words, when fermions are present, the factorization over the ideal $I$ may take away traces along with some traceless elements.

To summarize, as $o(M,2)$-module $hu(1|(1,2)\text{:}[M,2])$ decomposes into the sum of all finite dimensional $o(M,2)$-modules described by various traceless Young tableaux (2.21) except for those with the first column of heights 1 or $M+1$. (The trivial tableau with no cells and its dual described by the one-column tableau with $M+2$ cells are included). Each allowed irreducible $o(M,2)$-module appears in one copy.

The element

$$\Gamma = (i)^{\frac{1}{2}(M-2)(M-3)}\phi^0\phi^1\dots\phi^{M+1} \tag{2.25}$$

satisfies

$$\Gamma\phi^A = (-1)^{M+1}\phi^A\Gamma\,, \qquad \Gamma^2 = Id \tag{2.26}$$

and

$$\Gamma^\dagger = \Gamma\,. \tag{2.27}$$

As a result, the projectors

$$\Pi_\pm = \frac{1}{2}(1\pm\Gamma) \tag{2.28}$$

are hermitean

$$(\Pi_\pm)^\dagger = \Pi_\pm\,. \tag{2.29}$$

According to (2.26), the elements $\Gamma$ and $\Pi_\pm$ are central for odd $M$. As a result, analogously to the case of usual Clifford algebra, $hu(1|(1,2)\text{:}[M,2])$ decomposes for odd $M$ into direct sum of two subalgebras singled out by the projectors $\Pi_\pm$

$$hu(1|(1,2)\text{:}[M,2]) = hu^E(1|(1,2)\text{:}[M,2]) \oplus hu^E(1|(1,2)\text{:}[M,2])\,. \tag{2.30}$$

[5] By virtue of (2.24) one proves that a tensor twice dual to the original one in the two groups of anti-symmetrized indices must vanish because at least one of the metric tensors on the right hand side of (2.24) will be contracted with a pair of indices of the dualized tensor. By double dualization one gets back the original tensor which is therefore also zero.

Here $hu^E(1|(1,2){:}[M,2])$ is the subalgebra of $hu(1|(1,2){:}[M,2])$ spanned by the elements even in $\phi$, $f(Y,-\phi) = f(Y,\phi)$, described by various Young tableaux (2.21) with even numbers of cells. Note that for the particular case of $M = 3$, which corresponds to $AdS_4$, the algebra $hu^E(1|(1,2):[3,2])$ as a $o(M,2)$-module contains only rectangular two-row Young tableaux. As will be shown in section 5, in agreement with the $4d$ results of [4], this is the manifestation of the isomorphism $hu^E(1|(1,2){:}[3,2]) \sim hu(1|2{:}[3,2])$.

By definition of $hu^E(1|(1,2){:}[M,2])$, its elements commute with $\Gamma$ and $\Pi_\pm$. For even $M$ one can therefore define two algebras $hu^E_\pm(1|(1,2){:}[M,2])$ spanned by the elements of the form

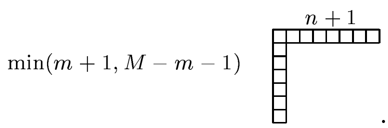
$$b \in hu^E_\pm(1|(1,2){:}[M,2]) : \qquad b = \Pi_\pm a\,, \qquad a \in hu^E(1|(1,2){:}[M,2])\,. \tag{2.31}$$

Note that $hu^E_\pm(1|(1,2){:}[M,2])$ are not subalgebras of $hu^E(1|(1,2){:}[M,2])$ because their elements do not satisfy (2.18) in the sector of the $osp(1,2)$ supercharges $t_j$. Elements of $hu^E_\pm(1|(1,2){:}[M,2])$ are even rank $o(M,2)$ tensors such that the tensors, described by the Young tableaux with the heights $p$ and $M+2-p$ of the first column, are dual to each other. In particular, the $o(M,2)$ generators in $hu^E_\pm(1|(1,2){:}[M,2])$ are $\Pi_\pm T_{AB}$ where $T_{AB}$ are the generators (2.3). For $M+2 = 4q$, the rank $2q$ $o(M,2)$ tensors are (anti)selfdual (for $M = 4q$, rank $2q+1$ tensors do not belong to $hu^E(1|(1,2){:}[M,2])$).

We will call the algebras $hu^E(1|(1,2){:}[M,2])$ and $hu^E_\pm(1|(1,2){:}[M,2])$ ($M$ is even) — type $A$ and type $B$ HS algebras, respectively. To summarize, let us list the gauge fields associated with the HS algebras defined in this section.

The gauge fields of $hu(1|2{:}[M,2])$ are 1-forms $\omega_{\underline{n}}^{A_1\ldots A_n,B_1\ldots B_n}$ carrying representations of $o(M,2)$ described by various two-row traceless rectangular Young tableaux of lengths $n = 0,1,2\ldots.$ As shown in [49, 2], these describe totally symmetric massless fields in $AdS_{M+1}$, i.e., the lowest energy subspace of the corresponding UIRREP of $o(M,2)$ is described by the rank $n+1$ totally symmetric traceless tensors of $o(M)$.

The gauge fields of $hu(1|(1,2){:}[M,2])$ are 1-forms $\omega_{\underline{n}}^{A_1\ldots A_n,B_1\ldots B_n,C_1\ldots C_m}$ carrying representations of $o(M,2)$ described by various traceless Young tableaux having two rows of equal length and one column of any height $m \le M$. There are two degenerate cases of 1-forms carrying totally antisymmetric representations of zero or maximal ranks $\omega_{\underline{n}}^{D_1\ldots D_n}$ with $n = 0$ or $M+2$ (while the cases with $n = 1$ or $n = M+1$ are excluded). According to the results of [41], the gauge fields $\omega_{\underline{n}}^{A_1\ldots A_n,B_1\ldots B_n,C_1\ldots C_m}$ with $n \ge 1$ describe massless fields in $AdS_{M+1}$ corresponding to the UIRREPs of $o(M,2)$ with the lowest energy states which form representations of $o(M)$ described by the traceless Young tableaux having one row of length $n+1$ and one column of height $\min(m+1, M-m-1)$




(2.32)

The degenerate cases of $o(M,2)$ singlet 1-forms $\omega_{\underline{n}}^{A_1\ldots A_n,B_1\ldots B_n,C_1\ldots C_m}$ with $n = m = 0$ and $n = 1$, $m = M$ correspond to two spin 1 fields. The fields with $n = 1, m = 0$ and $n = 1, m = M-2$ describe two graviton-like spin 2 fields.

The gauge fields of $hu^E(1|(1,2){:}[M,2])$ are 1-forms $\omega_{\underline{n}}^{A_1\dots A_n,B_1\dots B_n,C_1\dots C_m}$ with even $m$. The corresponding lowest energy representations of $o(M)\subset o(M,2)$ are described by the hook Young tableaux (2.32) with an odd number of cells in the first column.

The type $B$ chiral algebras $hu^E_\pm(1|(1,2){:}[M,2])$ ($M$ is even) give rise to gauge fields $\omega_{\underline{n}}^{A_1\dots A_n,B_1\dots B_n,C_1\dots C_m}$ with $m=l$ and $m=M-l-2$ related by the (anti)selfduality conditions. In particular, the fields with $m=2q$ for $M=4q+2$ are (anti)selfdual, i.e. the corresponding lowest energy representations of $o(4q+2)$ are described by (anti)selfdual Young tableaux with the first column of height $2q+1$. Note that type $B$ systems can be realized in odd-dimensional space-times $AdS_{M+1}$ which include the cases of $AdS_5$ and $AdS_{11}$ being of special interest from the superstring theory perspective.

## 3. Factorization by projection

To simplify the construction of HS algebras it is convenient to use the projection formalism analogous to that used in [49, 52] for the analysis of $5d$ HS models. The idea is that it is easy to factor out terms proportional to one or another set of operators $a_i$ if there is some element $\Delta$ such that $a_i\Delta=\Delta a_i=0$. Let $C$ be the centralizer of $a_i$, i.e., $f\in C:[a_i,f]=0$. Suppose that $\Delta$ also commutes with all elements of $C$, which is usually automatically true because $\Delta$ is in a certain sense built of $a_i$. Then elements $f\Delta=\Delta f$, $f\in C$ span $C/I$ where $I$ consists of such $g\in C$ that $g=g^l a_i$ or $g=a_i g^r$ for some $i$.

A little complication is that in many cases $\Delta$ does not belong to the original algebra because $\Delta^2$ does not exist (diverges). As a result, the space of elements $f\Delta=\Delta f$ forms a module of the original algebra rather than an algebra with respect to the original product law. However, one can redefine the product law of the elements of $g=f\Delta$ appropriately provided that there is an element $G$ such that

$$\Delta G\Delta=\Delta. \tag{3.1}$$

One simply defines

$$g_1\circ g_2=g_1 G g_2\,. \tag{3.2}$$

The new product is associative and reproduces the product law in $C/I$ so that

$$g_1\circ g_2=f_1f_2\Delta\,,\qquad g_i=f_i\Delta\,. \tag{3.3}$$

Note that $G$ is not uniquely defined because, with no effect on the final result, one can add to $G$ any terms $a_ip$ and $qa_i$ with $p$ and $q$ annihilating $\Delta w$. In fact, the role of $G$ is auxiliary because one can simply use (3.3) as a definition of the product law in $C/I$.

This situation can be illustrated by the example of the algebra of differential operators. Consider differential operators with polynomial coefficients of one variable $x$. Its generic element is $a(x,x')=\sum_{n,m=0}^\infty a_{n,m}x^n\delta^m(x-x')$ with a finite number of nonzero coefficients $a_{n,m}$ and

$$\delta^m(x-x')=\frac{\partial^m}{(\partial x)^m}\delta(x-x')\,. \tag{3.4}$$



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We consider simultaneously the case of usual commuting variable $x$ and the Grassmann case with $x^2 = 0$. (Recall that in the latter case $\delta(x) = x$.) The product law is defined as usual by $(ab)(x_1, x_2) = \int dx_3 a(x_1, x_3) b(x_3, x_2)$. Consider the subalgebra $C$ spanned by the elements which commute with $x$ (i.e. $x\delta(x - x')$). It is spanned by polynomials of $x$, i.e. $a \in C : a = \sum_{n=0}^{\infty} a_n x^n \delta(x - x')$. Obviously, the ideal $I$ is spanned by the elements proportional to $x$ which are various elements $a$ with zero constant term. The quotient algebra $C/I$ is spanned by constants $a_0 \delta(x - x')$.

Let us now do the same with the aid of projector. $\Delta$ is obviously the delta function

$$\Delta(x, x') = \delta(x)\delta(x') . \tag{3.5}$$

It satisfies $x\Delta = \Delta x = 0$. But $\Delta^2 = \Delta\delta(0)$ with $\delta(0) = 0$ in the Grassmann case and $\delta(0) = \infty$ in the commuting case. Note that this is not occasional but is a consequence of the original algebra properties. Actually, suppose that $\Delta^2$ is well-defined. Then $\Delta^2$ would satisfy the same properties $x\Delta^2 = \Delta^2 x = 0$ and one could expect that $\Delta^2 = \alpha\Delta$ with some coefficient $\alpha$. If $\alpha \neq 0$ or $\infty$, upon appropriate rescaling, $\Delta$ could be defined as a projector. However, this cannot be true because, formally, $\Delta^2 = \Delta[\frac{\partial}{\partial x}, x]\Delta = 0$. Therefore, either $\alpha = 0$ (Grassmann case) or $\Delta^2$ makes no sense (commuting case of $\alpha = \infty$). To redefine the product law according to (3.1), (3.2) one can set

$$G(x, x') = 1 . \tag{3.6}$$

There is an ambiguity in the choice of $G$. Any $G'(x, x') = G(x, x') + \sum_{n,m \geq 0} a_{n,m} x^n x'^m$ with $a_{0,0} = 0$ is equally good. Note that neither $\Delta$ nor $G$ belong to the original algebra of differential operators.

In the case of $hc(1|2{:}[M, 2])$ the operators $a_i$ identify with the generators $t_{ij}$ of $sp(2)$. Let us define the algebra of oscillators $Y_i^A$ as the star product algebra with the product law

$$f(Y) * g(Y) = (2\pi)^{-2(M+2)} \int d^{2(M+2)} S d^{2(M+2)} T \, f(Y + S) g(Y + T) \exp i S_j^A T_A^j , \tag{3.7}$$

which is the integral formula for the associative Weyl product (sometimes called Moyal product) of totally symmetrized products of oscillators. Here $Y_i^A$, $S_j^A$ and $T_j^A$ are usual commuting variables while the non-commutativity of the oscillator algebra results from the non-commutativity of the star product. From (3.7) the following relations follow

$$Y_j^A * g(Y) = \left( Y_j^A + i \frac{\partial}{\partial Y_A^j} \right) g(Y) , \qquad g(Y) * Y_j^A = \left( Y_j^A - i \frac{\partial}{\partial Y_A^j} \right) g(Y) . \tag{3.8}$$

To apply the projection method we need an element $\Delta$ that satisfies

$$t_{ij} * \Delta = \Delta * t_{ij} = 0 . \tag{3.9}$$

An appropriate ansatz is

$$\Delta = \Phi(z) , \qquad z = \frac{1}{4} Y_i^A Y_{Aj} Y^{Bi} Y_B^j , \tag{3.10}$$

where $z$ is both $sp(2)$ and $o(M,2)$ invariant and, therefore,

$$[\Delta, t_{ij}]_* = 0\,, \qquad [\Delta, T_{AB}]_* = 0\,, \tag{3.11}$$

where we use notations $[a,b]_* = a*b - b*a$, $\{a,b\}_* = a*b + b*a$. From (3.7) one finds that the condition $\{t_{ij}, \Delta\}_* = 0$ gives

$$\left( Y_i^A Y_{Aj} - \frac{\partial^2}{\partial Y^{Ai} \partial Y_A^j} \right) \Phi = 0\,. \tag{3.12}$$

It is elementary to check that it amounts to the differential equation

$$2z\Phi'' + (M+1)\Phi' - \Phi = 0\,, \tag{3.13}$$

where $\Phi'(z) = \frac{\partial}{\partial z}\Phi(z)$. For $M > 0$ this equation admits the unique solution analytic in $z$ which can be written in the form

$$\Phi(z) = \int_{-1}^{1} ds (1-s^2)^{\frac{1}{2}(M-2)} \exp s\sqrt{z}\,. \tag{3.14}$$

The analyticity in $z$ is because the integration region is compact and the measure is even under $s \to -s$ so that only even powers of $\sqrt{z}$ contribute. It is elementary to check that it satisfies (3.13) by partial integration over $s$.

Let $f \in hc(1|2{:}[M,2])$, i.e. $[f, t_{ij}]_* = 0$. According to the general scheme, $hu(1|2{:}[M,2])$ is spanned by the elements

$$f * \Delta = \Delta * f\,. \tag{3.15}$$

Note that $[\Delta, f]_* = 0$ because $\Delta$ is some function of $z$ (3.10) which itself is a function of $t_{ij}$.[6] It does not belong to the algebra since $\Delta * \Delta$ diverges. The product law in $hu(1|2{:}[M,2])$ is defined by the general formula (3.3).

The case of the algebra $hu(1|(1,2){:}[M,2])$ with Clifford generating elements $\phi^A$ can be considered analogously. First one replaces the Weyl star product (3.7) with the Weyl-Clifford one

$$\begin{aligned} f(Y,\phi) * g(Y,\phi) \;=\; & (2\pi)^{-2(M+2)} \int d^{2(M+2)}S d^{2(M+2)}T d^{M+2}\chi d^{M+2}\psi \exp(iS_j^A T_A^j - \chi^A \psi_A) \times \\ & \times f(Y+S, \phi+\chi) g(Y+T, \phi+\psi) \end{aligned} \tag{3.16}$$

such that the relations

$$\phi^A * X = \left(\phi^A - \frac{\partial}{\partial \phi_A}\right) X\,, \qquad X * \phi^A = X \left(\phi^A - \frac{\overleftarrow{\partial}}{\partial \phi_A}\right) \tag{3.17}$$

are true in addition to (3.8). Then one observes that the operator $\Delta_1$

$$\Delta_1 = \Phi(z_1)\,, \qquad z_1 = \frac{1}{4} Y_i^A Y_{Aj} Y^{Bi} Y_B^j + \frac{i}{2} Y_i^A \phi_A Y^{Bj} \phi_B \tag{3.18}$$

[6] It is useful to observe that any order $p$ polynomial of $z$ can be represented as some other order $p$ star polynomial of $t_{ij} * t^{ij}$. As a result, any polynomial of $z$ has zero star commutator with any $f \in hc(1|2{:}[M,2])$.

possesses the desired properties

$$t_i * \Delta_1 = \Delta_1 * t_i = 0\,, \tag{3.19}$$

$$t_{ij} * \Delta_1 = \Delta_1 * t_{ij} = 0\,. \tag{3.20}$$

The form of $\Delta_1$ can be guessed as follows. Since $\Delta_1$ is a Casimir operator of $osp(1,2)$ it is natural to expect that it is some function of its quadratic Casimir operator $z_1$ (note that the star commutators with $t_i$ and $t_{ij}$ have a form of some first-order differential operators so that their annulators form a ring). On the other hand, since $t_{ij}$ is $\phi$-independent, the $\phi$-independent part of the condition (3.20) implies that the functional dependence of $\Delta_1$ on $z_1$ must be the same as that of $\Delta(z)$ on $z$.

According to the general scheme, elements of $hu(1|2{:}[M,2])$ can be represented as

$$\Delta_1 * a = a * \Delta_1 \tag{3.21}$$

with various $a$ such that $[a, t_i]_* = 0$ (and, therefore, $[a, \Delta_1]_* = 0$). Now the factorization of elements $a = b^i * t_i$ or $a = t_i * b^i$ is automatic.

In applications it is convenient to use a slightly different realization of $\Delta_1$. Let us introduce the operator

$$L = -\frac{1}{4}(M-2) - \frac{i}{8}\phi^A\phi^B Y_{Aj} Y^j_B = 1 - \frac{i}{2} t_j * t^j\,. \tag{3.22}$$

It is easy to check that it has the following properties

$$L * t_i = \frac{1}{2} t_{ij} * t^j\,, \qquad t_i * L = -\frac{1}{2} t^j * t_{ij}\,, \tag{3.23}$$

$$L^2 = L - \frac{1}{8} t_{ij} * t^{ij}\,, \tag{3.24}$$

$$[t_{ij}, L]_* = 0\,. \tag{3.25}$$

From (3.24) it follows that $L$ is a projector modulo terms proportional to $t_{ij}$. In particular,

$$L^2 * \Delta = L * \Delta \qquad \Delta * L^2 = \Delta * L\,. \tag{3.26}$$

From (3.25) it follows that

$$[L, \Delta]_* = 0\,. \tag{3.27}$$

One observes that $L * \Delta$ has the same properties as $\Delta_1$. Indeed, using that $\Delta$ is annihilated by $t_{ij}$ from the left and from the right, one finds that $L * \Delta$ satisfies

$$t_i * (L * \Delta) = (\Delta * L) * t_i = 0\,, \qquad t_{ij} * (L * \Delta) = (L * \Delta) * t_{ij} = 0\,. \tag{3.28}$$

One therefore can represent $\Delta_1$ as $L * \Delta$. The precise relationship is

$$\Delta_1 = -\frac{4}{M-2} L * \Delta \tag{3.29}$$

as one can conclude from the first line in (3.22) by comparing the $\phi$ independent parts in $L * \Delta$ and $\Delta_1$. Let us note that the formula (3.29) does not work for $M = 2$. This means that the operators $\Delta_1$ and $L * \Delta$ are essentially different for this case. Presumably this is related to the fact that the algebra $o(2,2)$ is not simple, having two independent quadratic Casimir operators.

Since $L$ is a well-defined polynomial operator in the star product algebra, this allows us to give a useful alternative definition of the algebra $hu(1|(1,2){:}[M,2])$ as follows. Let $a(Y,\phi)$ be an arbitrary $sp(2)$ invariant element of the star product algebra

$$[a, t_{ij}]_* = 0\,. \tag{3.30}$$

Then $hu(1|(1,2){:}[M,2])$ is spanned by the elements

$$x = L * \Delta * a * L\,. \tag{3.31}$$

First note that because of (3.30) the same element $x$ can be equivalently written in any of the following forms

$$x = L * a * \Delta * L = L * a * L * \Delta = \Delta * L * a * L\,. \tag{3.32}$$

Using the identities (3.28) one proves that $t_i * x = x * t_i = 0$. Clearly, any two $a$ are equivalent if they differ by terms $t_i * b^i$ or $b^i * t_i$ with some $b^i$.

Let us now prove that the rank 1 and rank $M+1$ antisymmetric representations factor out of $hu(1|(1,2){:}[M,2])$

**Lemma.**

$$\Delta * L * \phi^A * L = 0\,, \qquad \Delta * L * \phi^{A_1} \dots \phi^{A_{M+1}} * L = 0\,. \tag{3.33}$$

*Proof.* The proof is elementary. One writes

$$\Delta * L * \phi^A * L = \Delta * L * \phi^A - \frac{i}{2} \Delta * L * \phi^A * t_j * t^j \tag{3.34}$$

and then observes that

$$i\Delta{*}L{*}\phi^A{*}t_j{*}t^j = i\Delta{*}L{*}\{\phi^A, t_j\}_*{*}t^j = -i\Delta{*}L{*}Y^A_j{*}t^j = -i\Delta{*}L{*}[Y^A_j, t^j]_* = 2\Delta{*}L{*}\phi^A\,. \tag{3.35}$$

The second identity in (3.33) follows from the first one along with $\Gamma * L = L * \Gamma$.

Now we are in a position to define simplest HS superalgebras in any dimension.

## 4. Higher spin superalgebras

To define a superalgebra which unifies $hu(1|2{:}[M,2])$ and $hu(1|(1,2){:}[M,2])$ we add two sets of elements $\chi_\mu$ and $\bar\chi^\mu$ which form conjugated spinor representations of the $o(M,2)$ Clifford algebra ($\mu, \nu \dots$ are spinor indices),

$$\chi_\mu * \phi^A = \gamma^A{}_\mu{}^\nu \chi_\nu\,, \qquad \phi^A * \bar\chi^\mu = \bar\chi^\nu \gamma^A{}_\nu{}^\mu\,, \tag{4.1}$$

where $o(M,2)$ gamma matrices $\gamma^A{}_\nu{}^\mu$ satisfy

$$\gamma^A{}_\nu{}^\rho \gamma^B{}_\rho{}^\mu + \gamma^B{}_\nu{}^\rho \gamma^A{}_\rho{}^\mu = -2\delta^\mu_\nu \eta^{AB}\,, \tag{4.2}$$

and commute with $Y_i^A$

$$\chi_\mu * Y_i^A = Y_i^A * \chi_\mu \,, \qquad \bar{\chi}^\mu * Y_i^A = Y_i^A * \bar{\chi}^\mu \,. \tag{4.3}$$

Introduce two projectors $\Pi_1$ and $\Pi_2$

$$\Pi_1 * \Pi_1 = \Pi_1 \,, \qquad \Pi_2 * \Pi_2 = \Pi_2 \,, \qquad \Pi_1 * \Pi_2 = \Pi_2 * \Pi_1 = 0 \,, \qquad \Pi_1 + \Pi_2 = I \tag{4.4}$$

and require

$$\Pi_1 * \chi_\mu = \chi_\mu * \Pi_2 = \chi_\mu \,, \qquad \Pi_2 * \bar{\chi}^\mu = \bar{\chi}^\mu * \Pi_1 = \bar{\chi}^\mu \,, \tag{4.5}$$

$$\Pi_1 * \phi^A = \phi^A * \Pi_1 = 0 \,, \qquad \Pi_2 * \phi^A = \phi^A * \Pi_2 = \phi^A \,, \qquad \{\phi^A, \phi^B\}_* = -2\eta^{AB}\Pi_2 \,. \tag{4.6}$$

As a result we have

$$\bar{\chi}^\mu * \bar{\chi}^\nu = 0 \,, \qquad \chi_\nu * \chi_\mu \ = 0 \,. \tag{4.7}$$

In addition we require

$$\chi_\nu * \bar{\chi}^\mu = -i\delta_\nu{}^\mu \Pi_1 \,, \tag{4.8}$$

and

$$\begin{aligned} \bar{\chi}^\mu * \chi_\nu &= -i\sigma^{-1}(M)(\overline{\exp}{-\phi_A\gamma^A})_\nu{}^\mu * \Pi_2 \\ &\equiv -i\sigma^{-1}(M) \sum_{p=0}^{p=M+2} (-1)^{\frac{p(p+1)}{2}} \frac{1}{p!} \phi^{A_1} \ldots \phi^{A_p} * \Pi_2 \gamma_{A_1 \ldots A_p \nu}{}^\mu \,, \end{aligned} \tag{4.9}$$

where $\sigma(M) = 2^{[M/2]+1}$ is the dimension of the spinor representation, i.e. $\mu, \nu = 1, \ldots, \sigma(M)$ and

$$\gamma_{A_1 \ldots A_p} = \gamma_{[A_1} \ldots \gamma_{A_p]} \equiv \gamma_{[[A_1} \ldots \gamma_{A_p]]} \tag{4.10}$$

form a basis of the Clifford algebra. $\overline{\exp}{-\phi_A\gamma^A}$ is defined as usual exponential with $\phi_A$ and $\gamma^A$ would be anticommuting to each other, i.e.,

$$\overline{\exp}\phi_A\gamma^A = \sum_{p=0}^{p=M+2} (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \phi^{A_1} \ldots \phi^{A_p} \gamma_{A_1 \ldots A_p} \,, \tag{4.11}$$

so that, taking into account (3.17), one gets in accordance with (4.1)

$$\phi^B * \overline{\exp}{-\phi_A\gamma^A} = \overline{\exp}{-\phi_A\gamma^A}\, \gamma^B \,, \qquad \overline{\exp}{-\phi_A\gamma^A} * \phi^B = \gamma^B\, \overline{\exp}{-\phi_A\gamma^A} \,. \tag{4.12}$$

The relative coefficients in (4.8) and (4.9) are fixed by the associativity of the spinor generating elements. (In practice it is enough to check that the two possible ways of computing $\bar{\chi}^\mu * \chi_\nu * \bar{\chi}^\nu * \chi_\mu$ give the same result.)

A general element of the HS superalgebra we call $hu(1,1|(0,1,2){:}[M,2])$ (the meaning of this notation is explained in the end of this section) is now represented in the form

$$a = \Delta * T * \Big( a_{11}(Y) * \Pi_1 + a_{22}(Y,\phi) * \Pi_2 + a_{12}^\mu(Y) * \chi_\mu + \bar{\chi}^\mu * a_{21\mu}(Y) \Big) * T \,, \tag{4.13}$$

where

$$T = \Pi_1 + \Pi_2 * L\,, \tag{4.14}$$

$a_{11}(Y)$, $a^{\mu}_{12}(Y)$ and $a_{21\mu}(Y)$ are some polynomials of $Y_i^A$, and $a_{22}(Y,\phi)$ is a polynomial of $Y_i^A$ and $\phi^A$, such that they all commute to the $sp(2)$ generators $t_{ij}$.

In fact, the projectors $\Pi_1$ and $\Pi_2$ are introduced to parametrize four blocks of a matrix which contains elements of the bosonic algebras $hu(1|2{:}[M,2])$ and $hu(1|(1,2){:}[M,2])$ in the diagonal blocks associated with the projectors $\Pi_1$ and $\Pi_2$, respectively, while odd elements of the HS superalgera are contained in the off-diagonal blocks. The coefficients $a_{11}$ and $a_{22}$ are assumed to be Grassmann even (commuting) while $a_{21\mu}$ and $a^{\mu}_{12}$ are Grassmann odd (anticommuting). This convention induces the superalgebra structure through the standard definition of field strengths with Grassmann odd spinor gauge fields. Note that the introduced projector structure can conveniently be described by the auxiliary Clifford variables $\theta*\theta=\bar\theta*\bar\theta=0, \{\theta,\bar\theta\}_*=1$, which have zero star commutators with all other generating elements. Then $\Pi_1=\theta*\bar\theta$, $\Pi_2=\bar\theta*\theta$, $\chi_\nu$ contains one power of $\theta$ and $\bar\chi^\mu$ contains one power of $\bar\theta$.

By construction, the elements $a^{\mu}_{12}(Y)$ of the form $\tilde a^{\nu}_{12}(Y)*Y_i^A\gamma_{A\nu}{}^{\mu}$ do not contribute to (4.13) as well as elements $a_{21\mu}$ of the form $\gamma_{A\nu}{}^{\mu}Y_i^A*\tilde a_{21\mu}(Y)$. As a result, representatives of the fermionic sectors of the superalgebra can be chosen in the form

$$a_{21\mu}(Y) = \sum_{n=0}^{\infty} a_{\mu}^{A_1\dots A_n,B_1\dots B_n} Y_{A_1}\dots Y_{A_n}\,Y_{B_1}\dots Y_{B_n}\,, \tag{4.15}$$

$$a^{\mu}_{12}(Y) = \sum_{n=0}^{\infty} \bar a^{\mu\,A_1\dots A_n,B_1\dots B_n} Y_{A_1}\dots Y_{A_n}\,Y_{B_1}\dots Y_{B_n}\,, \tag{4.16}$$

where the spinor-tensors $a_{\mu}^{A_1\dots A_n,B_1\dots B_n}$ and $\bar a^{\mu\,A_1\dots A_n,B_1\dots B_n}$ have symmetry properties of the two-row rectangular Young tableau with respect to the indices $A$ and $B$ (i.e. symmetrization over any $n+1$ indices gives zero) and satisfy the $\gamma$-transversality conditions

$$\gamma_{A_1\nu}{}^{\mu} a_{\mu}^{A_1\dots A_n,B_1\dots B_n} = 0\,, \qquad \bar a^{\mu\,A_1\dots A_n,B_1\dots B_n}\gamma_{A_1\,\mu}{}^{\nu} = 0\,. \tag{4.17}$$

As a result, the gauge fields associated with the superalgebra $hu(1,1|(0,1,2){:}[M,2])$ consist of bosonic and fermionic 1-forms. Bosonic gauge fields corresponding to the subalgebras $hu(1|2{:}[M,2])$ and $hu(1|(1,2){:}[M,2])$ are listed in the end of section 2. Fermionic fields $dx^{\underline{n}}\omega_{\underline{n}\mu}{}^{A_1\dots A_n,B_1\dots B_n}(x)$ and $dx^{\underline{n}}\bar\omega_{\underline{n}}{}^{\mu\,A_1\dots A_n,B_1\dots B_n}(x)$ belong to the two-row rectangular $\gamma$-transverse spinor-tensor representations of $o(d-1,2)$. These correspond to totally symmetric half-integer spin massless representations of $o(d-1,2)$.

The chiral superalgebras $hu_\pm(1,1|(0,1,2){:}[M,2])$ are obtained from $hu(1,1|(0,1,2){:}[M,2])$ with the aid of the projectors $\Pi_\pm$ (2.28)

$$f\in hu_\pm(1,1|(0,1,2){:}[M,2]):\qquad f=\Pi_\pm*g*\Pi_\pm\,,\qquad g\in hu(1,1|(0,1,2){:}[M,2])\,. \tag{4.18}$$

For even $M$ the projection (4.18) implies chiral projection for spinor generating elements and projects out bosonic elements which are odd in $\phi$. For odd $M$ this condition just implies irreducibility of the spinor representation of the Clifford algebra.

Now let us define a family of HS superalgebras $hu(n,m|(0,1,2)\text{:}[M,2])$ with non-abelian spin 1 subalgebras (i.e., Chan-Paton indices). To this end we consider the algebra of operator-valued $(n+m)\times(n+m)$ matrices of the form (4.13) such that the elements $a_{11} \to a_{11}{}^u{}_v$ are $n\times n$ matrices $(u,v=1,\dots n)$, elements $a_{22}\to a_{22}{}^{u'}{}_{v'}$ are $m\times m$ matrices $(u',v'=1,\dots m)$, elements $a_{12}\to a_{12}{}^u{}_{v'}$ are $n\times m$ matrices, and elements $a_{21}\to a_{21}{}^{u'}{}_v$ are $m\times n$ matrices.

The reality conditions which single out the appropriate real HS superalgebra $hu(n,m|(0,1,2)\text{:}[M,2])$ are

$$(a_{11}(Y))^{\dagger u}{}_v = -a_{11}(Y)^u{}_v\,, \qquad (a_{22}(Y))^{\dagger u'}{}_{v'} = -a_{22}(Y)^{u'}{}_{v'}\,, \tag{4.19}$$

and

$$(a_{12}^{\mu}(Y))^{\dagger u'}{}_v = a_{21\mu}{}^{u'}{}_v(Y)\,, \qquad (\chi_\mu)^\dagger = i\bar{\chi}^\mu\,, \tag{4.20}$$

where $\dagger$ denotes usual matrix hermitean conjugation along with the conjugation of the generating elements $Y_i^A$ and $\phi^A$ according to (2.2) and (2.23). It is easy to see that such conditions indeed single out a real form of the complex Lie superalgebra resulting from the original associative algebra with elements (4.13) by (anti)commutators as a product law. Note that $hu(n,m|(0,1,2)\text{:}[M,2])$ contains $u(n)\oplus u(m)$ as a finite dimensional subalgebra. The labels 0 and 1 in this notation indicate how many Clifford elements taking values in the vector representation of $o(M,2)$ appear in the respective diagonal blocks.

Analogously to the case of $4d$ HS algebras [4], the algebras $hu(n,m|(0,1,2)\text{:}[M,2])$ admit truncations by an antiautomorphism $\rho$ of the original associative algebra to the algebras $ho(n,m|(0,1,2)\text{:}[M,2])$ and $husp(n,m|(0,1,2)\text{:}[M,2])$ with $o(n)\oplus o(m)$ and $usp(n)\oplus usp(m)$ (here $n$ and $m$ are even) as finite-dimensional Yang-Mills (spin 1) subalgebras. The truncation condition is

$$\rho(a) = -i^{\pi(a)}a\,, \tag{4.21}$$

where $\pi(a)=0(1)$ for (even) odd elements of the superalgerba. The action of $\rho$ on the matrix indices is defined as usual

$$\rho(a_u{}^v) = -\rho^{vp}a_p{}^q\rho_{qu}\,, \quad \rho(a'_{u'}{}^{v'}) = -\rho^{v'p'}a'_{p'}{}^{q'}\rho_{q'u'}\,, \tag{4.22}$$

where the bilinear forms $\rho_{uv}$ and $\rho_{u'v'}$ are non-degenerate and either both symmetric (the case of $ho(n,m|(0,1,2)\text{:}[M,2])$) or antisymmetric (the case of $husp(n,m|(0,1,2)\text{:}[M,2])$). The action of $\rho$ on the generating elements is defined by the relations

$$\rho(Y_j^A) = iY_j^A \qquad \rho(\phi^A) = -\phi^A\,, \qquad \rho(\chi_\nu{}^u{}_{u'}) = \bar{\chi}^{\mu v'}{}_v\rho^{uv}\rho_{v'u'}C_{\mu\nu}\,, \tag{4.23}$$

where $C_{\mu\nu}$ is the charge conjugation matrix which represents the Clifford algebra antiautomorphism $\rho(\phi^A)$ in the chosen representation of $\gamma$ matrices, i.e.

$$\gamma^A{}_\nu{}^\mu = -C^{\mu\sigma}\gamma^A{}_\sigma{}^\eta C_{\eta\nu}\,, \qquad C^{\mu\sigma}C_{\nu\sigma} = \delta^\mu_\nu\,. \tag{4.24}$$

The chiral HS algebras $hu_{\pm}(n,m|(0,1,2){:}[M,2])$, $ho_{\pm}(n,m|(0,1,2){:}[M,2])$ and $husp_{\pm}(n,m,|(0,1,2){:}[M,2])$ are obtained from $hu(n,m|(0,1,2){:}[M,2])$, $ho(n,m|(0,1,2){:}[M,2])$ and $husp_{\pm}(n,m,|(0,1,2){:}[M,2])$ by the projection (4.18), taking into account that $\rho(\Gamma)=\Gamma$ and, therefore, $\rho(\Pi_{\pm})=\Pi_{\pm}$.

Finally let us note that the algebras $hu(n,m|(u,v,2){:}[M,2])$ with $u$ and $v$ copies of fermions in the upper and lower blocks are likely to be relevant to the analysis of HS gauge theories with mixed symmetry HS gauge fields. A number of copies of bosonic oscillators, which are assumed to be the same in the upper and lower blocks, can also be enlarged to the case of $hu(n,m,|(u,v,p){:}[M,2])$. In this notation $hu(1|2{:}[M,2])\sim hu(1,0|(0,0,2){:}[M,2])$ and $hu(1|(1,2){:}[M,2])\sim hu(1,0|(1,0,2){:}[M,2])$. More generally, (super)algebras with $u$ fermions and $p$ bosons in the upper block and $v$ fermions and $q$ bosons in the lower block (with all bosons and fermions taking values in the vector representation of $o(s,t)$) are denoted $hu(n,m,|(u,p;v,q){:}[s,t])$ (and similarly for the orthogonal ($ho$) and symplectic ($husp$) series). Note that these algebras are Lie superalgebras when $u+v$ is odd.

## 5. Spinorial realizations in lower dimensions

It is instructive to compare the HS superalgebras introduced in this paper with the HS superalgebras in lower space-time dimensions, defined earlier in terms of spinor oscillator algebras. For example, the simplest $AdS_4$ superalgebra $hu(1,1|4)$ (in notations of [4]) was realized [53, 54] as the algebra of functions $f(y,K)$ of the spinor oscillators $y_\mu$ and Klein operator $K$ satisfying relations

$$[y_\mu,y_\nu]_*=2iC_{\mu\nu}\,,\qquad K*y_\mu=-y_\mu*K\,,\qquad K^2=1\,, \tag{5.1}$$

where $C_{\mu\nu}=-C_{\nu\mu}$ is the $4d$ antisymmetric charge conjugation matrix. The spinorial generating elements $y_\mu$ are not subject to any further restrictions.

One can try to identify the generating elements $\chi_\mu$ and $\bar\chi^\mu$ with such independent spinorial generating elements. This does not work however for the general case because of the projectors $T$ in (4.13) which induce a nontrivial dependence on the bosonic and fermionic oscillators $\phi^A$ and $Y_i^A$ into $\chi_\mu$ and $\bar\chi^\mu$-dependent terms. To see what happens let us start with the case of general $M$.

First one observes that the operator

$$K=\Pi_1-\Pi_2\,,\qquad K*K=1 \tag{5.2}$$

behaves just as the Klein operator. From (4.5) it follows that it anticommutes with fermions. Let $C_{\mu\nu}$ be the charge conjugation matrix which is either symmetric or antisymmetric depending on $M$,

$$C_{\mu\nu}=(-1)^{\gamma(M)}C_{\nu\mu}\,. \tag{5.3}$$

Let us set

$$y_\mu=2T*\Big(\chi_\mu+\bar\chi^\nu C_{\nu\mu}\Big)*T\,. \tag{5.4}$$

(To simplify formulae, in this section we discard the operator $\Delta$ which is an overall factor in all expressions.) From (5.2) it follows that

$$K * y_\mu = 2T * \Big(\chi_\mu - \bar{\chi}^\nu C_{\nu\mu}\Big) * T\,. \tag{5.5}$$

As a result the set of elements $y_\mu$ and $K * y_\mu$ is as good as the original set of elements $\chi_\mu * L$ and $L * \bar{\chi}^\nu$ in the decomposition (4.13). From the defining relations (4.8) and (4.9) we obtain

$$\Pi_1 * y_\mu * y^\nu = 4\Pi_1 * \chi_\mu * L * \bar{\chi}^\nu\,, \tag{5.6}$$

$$\Pi_2 * y^\nu * y_\mu = 4\Pi_2 * L * \bar{\chi}^\nu * \chi_\mu * L\,. \tag{5.7}$$

These relations can be treated as defining relationships on spinor generating elements of the algebra. By means of (3.22), (4.1), (4.8), (4.9) they get an equivalent form

$$\Pi_1 * y_\mu * y^\nu = 4i\Pi_1 * \left(\frac{1}{4}(M-2)\delta_\mu^\nu + \frac{i}{8}\gamma^{AB}{}_\mu{}^\nu Y_{Aj}Y_B^j\right), \tag{5.8}$$

$$\Pi_2 * y^\nu * y_\mu = -4i\sigma^{-1}(M)\Pi_2 * L * \sum_{p=0}^{p=M+2}(-1)^{\frac{p(p+1)}{2}}\frac{1}{p!}\phi^{A_1}\dots\phi^{A_p} * L\gamma_{A_1\dots A_p\nu}{}^\mu\,. \tag{5.9}$$

The resulting expressions depend nontrivially on $\phi^A$ and $Y_i^A$ that means in particular that, generically, the anticommutator of the spinorial element $y_\mu$ gives rise to HS generators. Since $y_\mu$ and $Ky_\mu$ are the only candidates for usual supercharges, the expressions (5.8) and (5.9) indicate that the HS superalgebras under consideration possess no finite dimensional conventional SUSY subsuperalgebras for general $M$.

Let us now turn to the case of $M=3$ which corresponds to $AdS_4$. According to (2.30), the irreducibility in the case of odd $M$ implies that one has to work with the algebras $hu_\pm(1,1|(0,1,2){:}[M,2])$ (4.18) which implies that independent elements on the right hand side of (5.6) are even in the Clifford elements $\phi^A$. For $M=3$ the terms of zeroth, second and fourth order can appear. The key point is that, according to Lemma 3.1 the fourth order terms in $\phi$ do not contribute (factor out). As a result, only the terms containing $\delta_\nu{}^\mu$ and $\gamma_{AB\nu}{}^\mu$ survive. With lowered index $\mu$ these have opposite symmetry types. As a result, one obtains that the spinor variables $y_\mu$ do indeed satisfy the Heisenberg commutation relations (5.1). Clearly, this is not true for generic $M$ because fourth order and higher order terms in $\phi$ will contribute to the right hand side of the commutator of the spinor generating elements.

The anticommutator has the form

$$\{y_\mu, y_\nu\}_* = (\alpha\Pi_1 + \beta\Pi_2 * L) * T_{AB}\gamma^{AB}{}_{\mu\nu}\,, \tag{5.10}$$

with some nonzero $\alpha$ and $\beta$, where $T_{AB}$ is the $o(M,2)$ generator (2.15). Now one observes that the terms on the right hand side of (5.10) parametrize all $osp(1,2)$ invariant bilinear combinations of oscillators $Y_{Ai}$ and $\phi_A$ (in fact, this is the manifestation of the isomorphism $sp(4)\sim o(3,2)$). Taking into account that $sp(2)$ invariant polynomials of $Y_{Ai}$ and $osp(1,2)$ invariant polynomials in $Y_{Ai}$ and $\phi_A$ are star polynomials of the invariant bilinears $T^{AB}$, one concludes that $y_\nu$ and $K$ form an equivalent set of generating elements for the $M=3$ HS superalgebra, i.e. one can forget about the generating elements $Y_{Ai}$ and $\phi_A$ in the case of $M=3$, thus arriving at the purely spinorial realization of the $4d$ HS algebras originally introduced in [53, 54] and denoted $hu(1,1|4)$ in [4].

The analysis of $AdS_3$ [55, 56] HS algebras is analogous to that of the $4d$ case: because the terms of fourth order in $\phi^A$ do not appear, the spinor generating elements $y_\mu$ along with the Klein operator can be chosen as independent generating elements thus establishing isomorphism with the original spinor realization. Since the $AdS_3$ algebra is semisimple $o(2,2) \sim o(2,1) \oplus o(2,1)$, the corresponding HS extensions are also direct sums of the $M=1$ algebras. This fact manifests itself in the isomorphism $hu(1|2{:}[2,2]) \sim hu(1|2{:}[1,2]) \oplus hu(1|2{:}[1,2])$ which is not hard to prove by observing that any length $h$ rectangular two-row Young tableau of $o(2,2)$ decomposes into the direct sum of one selfdual and one anti-selfdual length $h$ rectangular two-row Young tableaux,[7] each forming a $o(2,1)$-module.

The isomorphisms between spinorial and vectorial realizations of $AdS_3$ and $AdS_4$ HS superalgebras extend to their matrix extensions $hu(n,m|2k)$, $ho(n,m|2k)$ and $husp(n,m|2k)$ considered originally in [4] for the spinorial realization and those considered in the end of the previous section for the vectorial realization. Namely,

$$hu(n,m|4) \sim hu_\pm(n,m|(0,1,2){:}[3,2])\,, \qquad hu(n,m|2) \sim hu_\pm(n,m|(0,1,2){:}[1,2])\,, \tag{5.11}$$

$$ho(n,m|4) \sim ho_\pm(n,m|(0,1,2){:}[3,2])\,, \qquad ho(n,m|2) \sim ho_\pm(n,m|(0,1,2){:}[1,2])\,, \tag{5.12}$$

$$husp(n,m|4) \sim husp_\pm(n,m|(0,1,2){:}[3,2])\,, \quad husp(n,m|2) \sim husp_\pm(n,m|(0,1,2){:}[1,2])\,. \tag{5.13}$$

From here it follows that usual $AdS_4$ superalgebras $osp(N,4)$ are subalgebras of the HS superalgebras with the parameters $n$ and $m$ high enough. Namely, one has for even $N$ [4]

$$osp(2p,2k) \subset hu(2^p,2^p|2k)\,, \tag{5.14}$$

$$osp(8p,2k) \subset ho(2^{4p-1},2^{4p-1}|2k)\,, \tag{5.15}$$

$$osp(8p+4,2k) \subset husp(2^{4p+1},2^{4p+1}|2k)\,. \tag{5.16}$$

Analogously, it follows also that $AdS_3$ SUSY algebras $osp(N_+,2) \oplus osp(N_-,2)$ are subalgebras of the appropriate $3d$ HS algebras of the form $h\cdots(n_+,m_+|2) \oplus h\cdots(n_-,m_-|2)$ where dots denote one of the three possible types of the algebra (unitary, orthogonal or unitary symplectic) and $n_\pm$ are appropriate $N_\pm$-dependent powers of two.

The case of $AdS_5$ corresponds to $M=4$. Here one takes the chiral HS algebra $hu_\pm(1,1|(0,1,2){:}[4,2])$. Again, in this case effectively only zero-order and second-order combinations of Clifford elements appear and it is possible to establish the isomorphism between the spinorial realization of the $AdS_5$ (equivalently $4d$ conformal) HS algebra and that given in this paper. Note that the spinorial realization of the $AdS_5$ HS algebra also

[7] As one can easily see, imposing opposite (anti)selfduality conditions on two pairs of tensor indices associated with different columns in a $o(2,2)$ Young tableau gives zero.

includes some reduction procedure which assumes a restriction to the centralizer of some operator $N$ followed by the factorization of the elements proportional to $N$ (see [57, 58, 16]). This does not allow one to express the spinor generating elements of the original $5d$ spinorial construction directly in terms of the generating elements of this paper. In fact it is this property which makes it impossible to extend this isomorphism to the case of $5d$ HS superalgebras which contain higher $N$-extended finite dimensional subsuperalgebras.

A simplest way to see this is by using the facts shown in the rest of this paper that the HS superalgebras considered here act on massless scalar and spinor singletons (i.e., boundary conformal fields). On the other hand, the conformal realization [16] of the spinorial $AdS_5$ HS algebras of [57, 58] deals with various $4d$ massless boundary supermultiplets which contain spins $s \geq 1$ for $N > 2$ . Thus, the maximal conventional $N$-extended $5d$ supersymmetry compatible with the HS algebras of the type considered in this paper is that with $N = 2$ associated with the boundary massless hypermultiplet. In notation of [16] the corresponding HS algebra is $husp_0(2,2|8)$ while its finite dimensional subsuperalgebra is $su(2,2|2)$. We therefore expect that, $su(2,2|2) \subset husp_\pm(2,2|(0,1,2){:}[4,2])$. The algebra $psu(2,2|4)$ associated with the $N = 4$ SYM supermultiplet is not a subalgebra of the HS algebras considered in this paper just because the $N = 4$ SYM supermultiplet contains spin one massless states absent in the scalar-spinor singleton realization of the HS algebras of this paper.

For analogous reasons we expect that the purely bosonic $7d$ HS algebra of [7] is isomorphic to $hu_\pm(1|2{:}[6,2])$ while the $M = 6$ HS superalgebras considered in this paper are all different from those discussed in [15] (which contain the finite dimensional subsuperalgebra denoted $osp(8^*|4)$ in [15]) because the latter are associated with the tensor singletons absent in the construction of this paper.

For $M > 6$, terms with higher combinations of the fermionic oscillators and $\gamma$ matrices appear in the defining relations for spinorial elements $y_\mu$ that complicates the spinorial realization of the HS algebras of this paper. Note, however, that the superalgebras with unrestricted spinorial elements suggested in [60] may still be relevant for the description of HS theories with mixed symmetry HS gauge fields in higher dimensions.

## 6. Scalar conformal module

According to notations of [1], $Rac$ is the unitary representation of $o(3,2)$ realized by a $3d$ conformal massless scalar field. The global conformal HS symmetry of a massless scalar in $M$-dimensions $hu(1|2{:}[M,2])$ (more precisely, its complexification) was originally introduced by Eastwood in [32]. For our purpose it is most convenient to use its realization in terms of bosonic oscillators as explained in section 2.

Let us introduce mutually conjugated oscillators

$$a^A = \frac{1}{2}(Y_1^A - iY_2^A)\,, \qquad \bar{a}^A = \frac{1}{2}(Y_1^A + iY_2^A)\,, \qquad (a^A)^\dagger = \bar{a}^A\,, \tag{6.1}$$

which satisfy the commutation relations

$$[a^A, \bar{a}^B] = -\eta^{AB}\,, \qquad [a^A, a^B] = 0\,, \qquad [\bar{a}^A, \bar{a}^B] = 0\,. \tag{6.2}$$

For the space-like values of $A = a = 1\dots M$ with $\eta^{ab} = -\delta^{ab}$ these are normal commutation relations for creation and annihilation operators. For the time-like directions $A = 0, M+1$ it is convenient to introduce the set of oscillators

$$\bar{\alpha} = a^0 + ia^{M+1}\,, \qquad \bar{\beta} = a^0 - ia^{M+1}\,, \qquad \alpha = \bar{a}^0 - i\bar{a}^{M+1}\,, \qquad \beta = \bar{a}^0 + i\bar{a}^{M+1} \tag{6.3}$$

($\alpha^\dagger = \bar{\alpha}, \beta^\dagger = \bar{\beta}$), which have the nonzero commutation relations

$$[\alpha, \bar{\alpha}] = 2\,, \qquad [\beta, \bar{\beta}] = 2\,. \tag{6.4}$$

The generators of the algebra $o(M,2)$ are

$$T^{AB} = -(\bar{a}^A a^B - \bar{a}^B a^A)\,. \tag{6.5}$$

The $AdS_{M+1}$ energy operator (1.2) is

$$E = \frac{1}{2}(\bar{\alpha}\alpha - \bar{\beta}\beta)\,. \tag{6.6}$$

The noncompact generators of $o(M,2)$ are

$$T^{+a} = -i(\beta a^a - \bar{\alpha}\bar{a}^a)\,, \qquad T^{-a} = i(\bar{\beta}\bar{a}^a - \alpha a^a)\,. \tag{6.7}$$

Let us now introduce the Fock module $U$ of states $|\Psi\rangle = \psi(\bar{a}, \bar{\alpha}, \bar{\beta})|0\rangle$ induced from the vacuum state

$$a^a|0\rangle = 0\,, \qquad \alpha|0\rangle = 0\,, \qquad \beta|0\rangle = 0\,. \tag{6.8}$$

$U$ is endowed with the positive definite norm with respect to which $a^A$ is conjugated to $\bar{a}^A$, i.e. the conjugated vacuum $\langle 0|$ is defined by

$$\langle 0|\bar{a}^a = 0\,, \qquad \langle 0|\bar{\alpha} = 0\,, \qquad \langle 0|\bar{\beta} = 0\,. \tag{6.9}$$

Let us note that the vacuum vector $|0\rangle$ is not annihilated by $T^{-a}$ in contrast with the standard singleton construction in terms of spinorial oscillators used for the description of singleton and doubleton modules [8]. We consider the singleton submodule $S \subset U$ spanned by the $sp(2)$ invariant states satisfying $t_{ij}|\Psi\rangle = 0$. These conditions are equivalent to

$$\tau^-|\Psi\rangle = \tau^+|\Psi\rangle = \tau^0|\Psi\rangle = 0\,, \tag{6.10}$$

where

$$\tau^- = a_a a^a + \bar{\alpha}\bar{\beta}\,, \qquad \tau^+ = \bar{a}_a\bar{a}^a + \alpha\beta\,, \qquad \tau^0 = \{a_a, \bar{a}^a\} + \frac{1}{2}\{\alpha, \bar{\alpha}\} + \frac{1}{2}\{\beta, \bar{\beta}\} \tag{6.11}$$

(note that $a_a b^a = a_a b_b \eta^{ab}$ with $\eta^{ab} = -\delta^{ab}$.) By its definition, $S$ forms a $hu(1|2{:}[M,2])$-module. Let us note that the conditions (6.10) are consistent because the $o(M,2)$ invariant metric has signature $(M,2)$. In the case of euclidean signature, for example, the conditions (6.10) associated with $\tau_0$ and $\tau^+$ would allow no solution at all in a unitary module. Note that, although some details of the realization of appropriate modules are different, our construction is a version of that of the two-time formalism developed by Marnelius and Nilsson [61] and Bars and collaborators [62], in which the $sp(2)$ invariance condition also plays the key role.

The generic solution of the conditions (6.10) is

$$
\begin{aligned}
|\Psi(\bar a,\bar\alpha,\bar\beta)\rangle &= |\Psi^+(\bar a,\bar\alpha,\bar\beta)\rangle + |\Psi^-(\bar a,\bar\alpha,\bar\beta)\rangle\,, && (6.12)\\
|\Psi^\pm(\bar a,\bar\alpha,\bar\beta)\rangle &= \psi^\pm(\bar a,\bar\alpha,\bar\beta)|0\rangle\,, && (6.13)
\end{aligned}
$$

where

$$
\begin{aligned}
\psi^+(\bar a,\bar\alpha,\bar\beta) &= \sum_{p,n=0}^{\infty}\frac{(-1)^p}{2^{2p}p!(p+n+\frac12 M-1)!}(\bar a_a\bar a^a)^p\bar\alpha^{p+n+\frac12 M-1}\bar\beta^p\psi_n^+(\bar a)\,, && (6.14)\\
\psi^-(\bar a,\bar\alpha,\bar\beta) &= \sum_{p,n=0}^{\infty}\frac{(-1)^p}{2^{2p}p!(p+n+\frac12 M-1)!}(\bar a_a\bar a^a)^p\bar\beta^{p+n+\frac12 M-1}\bar\alpha^p\psi_n^-(\bar a) && (6.15)
\end{aligned}
$$

and $\psi_n^\pm(\bar a)$ are arbitrary degree $n$ harmonic polynomials, i.e.

$$
\psi_n^\pm(\bar a)=\psi^\pm_{a_1\cdots a_n}\bar a^{a_1}\cdots\bar a^{a_n}\,,\qquad \psi^\pm_{bca_3\ldots a_n}\eta^{bc}=0\,. \tag{6.16}
$$

Let us note that the Fock space realization of the modules $|\Psi^\pm\rangle=\psi^\pm|0\rangle$ is literally valid for even $M$ when all powers of oscillators are integer. Nevertheless, the modules $|\Psi^\pm\rangle$ are well-defined for odd $M$ as well. Actually, although powers of one of the oscillators $\bar\alpha$ or $\bar\beta$ are half-integer for odd $M$, the modules $|\Psi^\pm\rangle$ are semi-infinite because the powers of the another oscillator ($\bar\beta$ or $\bar\alpha$) are nonnegative integers. We therefore will use the formulae (6.14) and (6.15) for all $M$.

The expressions (6.14) and (6.15) admit the following useful form

$$
|\Psi^\pm\rangle=\bar{\mathcal P}^\pm\psi^\pm(\bar a)|0\rangle\,, \tag{6.17}
$$

where $\psi^\pm(\bar a)$ are the harmonic polynomials (6.16) and

$$
\begin{aligned}
\bar{\mathcal P}^+ &= \oint d\mu\exp\left(\mu^{-1}\bar\alpha-\frac14\mu\bar a^a\bar a_a\bar\beta\right)\mu^{-\frac12\{a^b,\bar a_b\}-2}\,, && (6.18)\\
\bar{\mathcal P}^- &= \oint d\mu\exp\left(\mu^{-1}\bar\beta-\frac14\mu\bar a^a\bar a_a\bar\alpha\right)\mu^{-\frac12\{a^b,\bar a_b\}-2}\,, && (6.19)
\end{aligned}
$$

where the integral is defined by

$$
\oint d\mu\mu^{-p}=\delta(p-1)\,,\qquad \delta(p)=1(0)\,,\qquad p=0(\neq 0)\,. \tag{6.20}
$$

Using this representation it is easy to check that (6.10) is true. Let us note that the operators $\bar{\mathcal P}^\pm$ cannot be expanded in power series of the oscillators since they effectively contain terms of the type $\alpha^{-\frac12\{a^b,\bar a_b\}-1}$ or $\beta^{-\frac12\{a^b,\bar a_b\}-1}$.

The Fock space norm of the states $|\Psi^\pm\rangle$ diverges. Indeed it is easy to see that $\langle\bar\Psi^+|\Psi^+\rangle\sim\sum_{p=0}^\infty 1$ using that

$$
\langle\bar\Psi_n|(a_aa^a)^p(\bar a_a\bar a^a)^p|\bar\Psi_n\rangle=2^{2p}\frac{p!(\frac12 M+p+n-1)!}{(\frac12 M+n-1)!}\,, \tag{6.21}
$$

where $n$ refers to a power of $\psi_n^+$ (6.16). This fact is neither occasional nor problematic, being a manifestation of the standard inner product problem with the Dirac quantization prescription for first-class constraints. Indeed, here the "first class constraints" (6.10) form the Lie algebra $sp(2)$. For normalizable states annihilated by the first-class constraints $\tau^i|\Psi_{1,2}\rangle = 0$ one obtains

$$\langle\bar{\Psi}_1|(\tau^i\chi_1 + \chi_2\tau^i)|\Psi_2\rangle = 0 \tag{6.22}$$

for any $\chi_{1,2}$. Choosing appropriately $\chi_{1,2}$ one finds a contradiction with the assumption that the corresponding matrix element is finite. For example, choosing $\chi_1 = -\chi_2 = \alpha\beta$ one finds for $\tau^-$ that all matrix elements of the manifestly positive-definite operator $\{\alpha, \bar{\alpha}\} + \{\beta, \bar{\beta}\}$ must vanish, that means that they cannot be finite. Note that this phenomenon has the same origin as the fact of non-existence of $\Delta^2$ for a projection-like operator $\Delta$ of section 3. A way out is also analogous.

A well-defined scalar product $\langle\langle | \rangle\rangle$ for the $sp(2)$ invariant module must break down the $sp(2)$ invariance of the Fock scalar product. This is achieved by redefining the scalar product in the form

$$\langle\langle\bar{\Psi}_1|\Psi_2\rangle\rangle = \langle\bar{\Psi}_1|G|\Psi_2\rangle\,, \tag{6.23}$$

where $G$ is some operator which does not commute with the $sp(2)$ generators. We demand the scalar product $\langle\langle\bar{\Psi}_1|\Psi_2\rangle\rangle$ to induce an $o(M,2)$ invariant norm on the $sp(2)$ invariant states. This can be achieved by choosing such $G$ that

$$[T^{AB}, G] = [\tau^{ij}, X_{ij}^{AB}] \tag{6.24}$$

for some $X_{ij}^{AB}$.

The appropriate operators $G^\pm$ for the modules $|\Psi^\pm\rangle$ are

$$G^+ = (\bar{\alpha}\alpha + 2)^{-1}F^+\,, \qquad G^- = (\bar{\beta}\beta + 2)^{-1}F^-\,, \tag{6.25}$$

where $F^+$ and $F^-$ are the Fock projectors for the oscillators $\beta, \bar{\beta}$ and $\alpha, \bar{\alpha}$ respectively,

$$\beta F^+ = 0\,, \qquad F^+\bar{\beta} = 0\,, \qquad [\alpha\,, F^+] = 0\,, \qquad [\bar{\alpha}\,, F^+] = 0\,, \tag{6.26}$$

$$\alpha F^- = 0\,, \qquad F^-\bar{\alpha} = 0\,, \qquad [\beta\,, F^-] = 0\,, \qquad [\bar{\beta}\,, F^-] = 0\,, \tag{6.27}$$

$$[a_a\,, F^\pm] = 0\,, \qquad [\bar{a}_a\,, F^\pm] = 0\,, \tag{6.28}$$

which are normalized so that

$$\langle 0|F^\pm|0\rangle = 1\,. \tag{6.29}$$

The Fock projectors $F^\pm$ can be realized as

$$F^+ = \sum_{n,m=0}^{\infty} \frac{(-1)^n}{2^m n! m!} \bar{a}_{a_1} \dots \bar{a}_{a_n} (\bar{\alpha})^m |0\rangle\langle 0| a^{a_1} \dots a^{a_n} (\alpha)^m\,, \tag{6.30}$$

$$F^- = \sum_{nm} \frac{(-1)^n}{2^m n! m!} \bar{a}_{a_1} \dots \bar{a}_{a_n} (\bar{\beta})^m |0\rangle\langle 0| a^{a_1} \dots a^{a_n} (\beta)^m\,. \tag{6.31}$$

The scalar products

$$\langle\langle\bar{\Psi}_1|\Psi_2\rangle\rangle_\pm = \langle\bar{\Psi}_1^\pm|G^\pm|\Psi_2^\pm\rangle \tag{6.32}$$

are $o(M,2)$ invariant. This property is manifest for the generators $T^{ab}$ and $E$, respectively. For the noncompact generators $T^{\pm a}$ this is also true because

$$[T^{+a}, G^+] = i[\tau^+, \bar{\alpha} a^a (\bar{\alpha}\alpha\alpha\bar{\alpha})^{-1} F^+]\,, \qquad [T^{-a}, G^+] = i[\tau^-, \alpha \bar{a}^a (\bar{\alpha}\alpha\alpha\bar{\alpha})^{-1} F^+]\,, \quad (6.33)$$

$$[T^{+a}, G^-] = i[\tau^+, \bar{\beta} a^a (\bar{\beta}\beta\beta\bar{\beta})^{-1} F^-]\,, \qquad [T^{-a}, G^-] = i[\tau^-, \beta \bar{a}^a (\bar{\beta}\beta\beta\bar{\beta})^{-1} F^-]\,, \quad (6.34)$$

as can be seen using (6.7). Note that the operators $\alpha\bar{\alpha}$ and $\bar{\alpha}\alpha\alpha\bar{\alpha}$ are positive definite and therefore their inverse operators are well defined.

In terms of components (6.16), the scalar product (6.23) gets the manifestly positive-definite form

$$\langle\langle \bar{\Psi}_1^{\pm} | \Psi_2^{\pm} \rangle\rangle = \sum_{n=0}^{\infty} \frac{2^{n+\frac{1}{2}M-2} n!}{(n+\frac{1}{2}M)!} \bar{\psi}^{\pm}_{n\, a_1 \ldots a_n} \psi_n^{\pm a_1 \ldots a_n}\,. \quad (6.35)$$

As a result, $|\Psi^{\pm}\rangle$ form unitary $o(M,2)$-modules.

The modules $|\Psi^+\rangle$ and $|\Psi^-\rangle$ have, respectively, positive and negative energies

$$E_n^+ = n + \frac{1}{2}M - 1 \qquad n = 0,1,2\ldots\,, \quad (6.36)$$

$$E_n^- = -n - \frac{1}{2}M + 1 \qquad n = 0,1,2\ldots\,. \quad (6.37)$$

The lowest energy of $|\Psi^+\rangle$ therefore is

$$E_0^+ = \frac{1}{2}M - 1\,. \quad (6.38)$$

This is the correct value for the conformal scalar in $M$ dimensions.[8] From (6.7) it follows that the lowest energy state of $|\Psi^+\rangle$ annihilated by $T^{-a}$ is that with the $\bar{a}$ independent part of $\psi_0^+(\bar{a})$ in (6.14), which is $o(M)$ singlet. From (1.6) one finds that the value of the $o(M,2)$ Casimir operator for the module $|\Psi^+\rangle$ is (see also [62])

$$C_2 = -\frac{1}{4}(M^2 - 4)\,. \quad (6.39)$$

Taking into account that $|\Psi^+\rangle$ is $sp(2)$ singlet, this is in agreement with (2.7).

Let us note that representation of the generators $T^{-a}$ and $E$ on the modes $\psi_n(\bar{a})$ has the following simple form

$$T^{-a} |\Psi^+(\psi(\bar{a}))\rangle = \left| \Psi^+ \left( i \frac{\partial}{\partial \bar{a}_a} \psi(\bar{a}) \right) \right\rangle, \quad (6.40)$$

$$E |\Psi^+(\psi(\bar{a}))\rangle = \left| \Psi^+ \left( (\bar{a}_a \frac{\partial}{\partial \bar{a}_a} + \frac{1}{2}M - 1) \psi(\bar{a}) \right) \right\rangle. \quad (6.41)$$

[8] Note that the operator $E$ here is some combination of $P^0$ and $K^0$ in the conformal algebra. The value (6.38) coincides with the scaling dimension for the complex equivalent scalar field conformal module.

## 7. Spinor conformal module

According to notations of [1], $Di$ is the unitary representation of $o(3,2)$ realized by $3d$ conformal massless spinor field. The global HS symmetry algebra which is the $M$ dimensional conformal HS symmetry of a massless spinor is $hu(1|(1,2){:}[M,2])$.

Consider the Fock module generated from the vacuum state satisfying

$$a^a|0\rangle_{\pm\alpha} = 0\,, \qquad \alpha|0\rangle_{\pm\alpha} = 0\,, \qquad \beta|0\rangle_{\pm\alpha} = 0\,, \qquad \phi|0\rangle_{\pm\alpha} = 0\,, \tag{7.1}$$

$$\phi_a|0\rangle_{\pm\alpha} = |0\rangle_{\pm\beta}\gamma_a{}^\beta{}_\alpha\,, \qquad |0\rangle_{\pm\beta}\gamma^\beta{}_\alpha = \pm|0\rangle_{\pm\alpha}\,, \tag{7.2}$$

where

$$\bar\phi = \frac{1}{2}(\phi_0 - i\phi_{M+1})\,, \quad \phi = \frac{1}{2}(\phi_0 + i\phi_{M+1})\,, \quad \phi^2 = 0\,, \quad \bar\phi^2 = 0\,, \quad \{\bar\phi,\phi\} = -1\,, \tag{7.3}$$

and the $\gamma$-matrices form some representation of the Clifford algebra $C_M$ associated with the compact algebra $o(M) \subset o(M,2)$

$$\{\gamma_a\,,\gamma_b\} = -2\eta_{ab}\,, \qquad \gamma = (i)^{\frac{1}{2}M(M-1)}\gamma_1\ldots\gamma_M\,. \tag{7.4}$$

In other words $|0\rangle_{\pm\alpha}$ forms a spinor representation of $o(M)$. Here $\alpha$ is the $o(M)$ spinor index while $\pm$ distinguishes between left and right spinors. A general element of the Fock module is

$$|\Psi\rangle_\pm = \sum_{m,n}^{\infty}\sum_{s=0}^{1} A(m,n,s)(\bar a_a\phi^a)^m\bar\alpha^{r(m,n,s)}\bar\beta^{q(m,n,s)}\bar\phi^s|\psi_n(\bar a)\rangle_\pm\,, \tag{7.5}$$

where

$$\phi^a a_a|\psi_n(\bar a)\rangle_\pm = 0\,, \qquad \bar a^a a_a|\psi_n(\bar a)\rangle_\pm = -n|\psi_n(\bar a)\rangle_\pm\,, \tag{7.6}$$

i.e.

$$|\psi_n(\bar a)\rangle_\pm = \psi^{a_1\ldots a_n\,\alpha}\bar a_{a_1}\ldots\bar a_{a_n}|0\rangle_{\pm\alpha} \tag{7.7}$$

is the generating function for rank $n$ totally symmetric tensor-spinors $\psi^{a_1\ldots a_n\,\alpha}$ satisfying the $\gamma$-transversality condition

$$\gamma_{a_1}{}^\alpha{}_\beta\psi^{a_1\ldots a_n\,\beta} = 0\,. \tag{7.8}$$

The supergenerators of $osp(1,2)$ are

$$Q = a^a\phi_a + \bar\alpha\bar\phi + \bar\beta\phi\,, \qquad \bar Q = \bar a^a\phi_a + \alpha\phi + \beta\bar\phi\,. \tag{7.9}$$

Imposing the $osp(1,2)$ invariance condition

$$Q|\Psi\rangle_\pm = 0\,, \qquad \bar Q|\Psi\rangle_\pm = 0 \tag{7.10}$$

we single out a $hu(1|(1,2){:}[M,2])$-module. This leads to a set of conditions on the parameters of (7.5) which admit the following general solution

$$q(m,n,s) = \frac{1}{2}\,(m - s + (2n+M-1)|m+s+1|_2)\,, \tag{7.11}$$

$$r(m,n,s) = \frac{1}{2}\,(m + s - 1 + (2n+M-1)|m+s|_2)\,, \tag{7.12}$$

$$A(m,n,s) = (-1)^{\frac{1}{2}m(m+1)}\frac{A(n,|m+s|_2)}{(m-|m|_2)!!(m+2n+M-1-|m+1|_2)!!}\,, \tag{7.13}$$

where $A(n,|m+s|_2)$ are arbitrary coefficients and we use notations

$$|2k|_2 = 0\,, \qquad |2k+1|_2 = 1\,. \tag{7.14}$$

The ambiguity in the coefficients $A(n,|m+s|_2)$ manifests the freedom of normalization of the spinor-tensors $|\psi_n\rangle_\pm$ (the dependence on $n$) and the fact that the module as a whole decomposes into the direct sum of two submodules spanned by the vectors which are odd and even in $\phi$ (the dependence on $|m+s|_2$). As a result,

$$|\Psi\rangle_\pm = |\Psi^+\rangle_\pm + |\Psi^-\rangle_\pm\,, \tag{7.15}$$

where, fixing appropriately $A(n,|m+s|_2)$,

$$\begin{aligned} |\Psi^+\rangle_\pm \,=\, & \sum_{p,n=0}^{\infty}\sum_{s=0,1}\frac{(-1)^{p+s}}{2^{2p+s}p!(p+n+\frac{1}{2}M+s-1)!}\times \\ & \times(\bar a_a\bar a^a)^p\bar\alpha^{p+n+\frac{1}{2}M+s-1}\bar\beta^p(\bar a_a\phi^a)^s\bar\phi^s|\psi^+_n(\bar a)\rangle_\pm\,, \end{aligned} \tag{7.16}$$

$$\begin{aligned} |\Psi^-\rangle_\pm \,=\, & \sum_{p,n=0}^{\infty}\sum_{s=0,1}\frac{(-1)^{p+s}}{2^{2p+s}p!(p+n+\frac{1}{2}M+s-1)!}\times \\ & \times(\bar a_a\bar a^a)^p\bar\beta^{p+n+\frac{1}{2}M+s-1}\bar\alpha^p(\bar a_a\phi^a)^s\phi^s\bar\phi|\psi^-_n(\bar a)\rangle_\pm\,. \end{aligned} \tag{7.17}$$

Although the modules $|\Psi^\pm\rangle_\pm$ do not belong to the original Fock module for odd $M$ because a power of the oscillator $\bar\alpha$ or $\bar\beta$ may be half-integer, they are well-defined semi-infinite modules because one of the powers is necessarily integer. From (7.2) it follows that the modules $|\Psi^\pm\rangle_\pm$ have definite chirality

$$\Gamma|\Psi^{\cdots}\rangle_\pm = \pm|\Psi^{\cdots}\rangle_\pm \tag{7.18}$$

in agreement with the fact that $\Gamma$ leaves invariant the space of $osp(1,2)$ singlets.

An analogue of the formula (6.17) is

$$|\Psi^\pm\rangle_{\dots} = \bar{\mathcal{P}}^\pm|\psi^\pm(\bar a)\rangle_{\dots}\,, \tag{7.19}$$

$$\bar{\mathcal{P}}^+ = \oint d\mu\exp\left(\mu^{-1}\bar\alpha-\mu\left(\frac{1}{4}\bar a^a\bar a_a\bar\beta+\frac{1}{2}a_a\phi^a\bar\phi\right)\right)\mu^{-\frac{1}{2}\{a^b,\bar a_b\}-2}\,, \tag{7.20}$$

$$\bar{\mathcal{P}}^- = \oint d\mu\exp\left(\mu^{-1}\bar\beta-\mu\left(\frac{1}{4}\bar a^a\bar a_a\bar\alpha+\frac{1}{2}a_a\phi^a\phi\right)\right)\mu^{-\frac{1}{2}\{a^b,\bar a_b\}-2}\bar\phi\,. \tag{7.21}$$

Note that the additional factor of $\bar\phi$ in the expression for $\bar{\mathcal{P}}^-$ maps the states $|\psi^\pm(\bar a)\rangle_\pm$ annihilated by $\phi$ to those annihilated by $\bar\phi$.

The $AdS_{M+1}$ energy operator is

$$E = \frac{1}{2}(\bar\alpha\alpha-\bar\beta\beta-[\phi,\bar\phi])\,. \tag{7.22}$$

As a result, the modules $|\Psi^\pm\rangle_\pm$ have energies

$$E^+_n = n+\frac{1}{2}(M-1) \tag{7.23}$$

and

$$E_n^- = -n - \frac{1}{2}(M-1)\,, \tag{7.24}$$

respectively, depending on the upper sign $+$ or $-$ on $\Psi$. The lowest energy of $|\Psi^+\rangle_\pm$ therefore is

$$E_0^+ = \frac{1}{2}(M-1) \tag{7.25}$$

that is the correct value for the conformal spinor in $M$ dimensions equal to its canonical scaling dimension.

The $o(M,2)$ invariant norm is defined by the same formulas (6.25)–(6.32). The resulting expression

$$\langle\langle \bar{\Psi}_1^\pm | \Psi_2^\pm \rangle\rangle = \sum_{n=0}^{\infty} 2^{n+\frac{1}{2}M-2} \frac{1}{(n+\frac{1}{2}M+1)!} \langle \bar{\psi}_n^\pm(a) | \psi_n^\pm(\bar{a}) \rangle\,, \tag{7.26}$$

is manifestly positive-definite. Thus the modules $|\Psi^\pm\rangle_\pm$ are unitary.

The noncompact generators are

$$T^{+a} = -i(\beta a^a - \bar{\alpha}\bar{a}^a + \phi\phi^a)\,, \qquad T^{-a} = i(\bar{\beta}\bar{a}^a - \alpha a^a - \bar{\phi}\bar{\phi}^a)\,. \tag{7.27}$$

The action of the generators $T^{-a}$ and $E$ has the form

$$T^{-a}|\Psi^+(\psi(\bar{a}))\rangle_\pm = \left|\Psi^+\left(i\frac{\partial}{\partial \bar{a}_a}\psi(\bar{a})\right)\right\rangle_\pm\,, \tag{7.28}$$

$$E|\Psi^+(\psi(\bar{a}))\rangle_\pm = \left|\Psi^+\left(\left(\bar{a}_a\frac{\partial}{\partial \bar{a}_a} + \frac{1}{2}(M-1)\right)\psi(\bar{a})\right)\right\rangle_\pm\,. \tag{7.29}$$

## 8. Generalized Flato-Fronsdal theorem

An important observation by Flato and Fronsdal [1] was that the tensor product of a pair of $AdS_4$ Dirac singletons [43] identified with the $3d$ massless particles gives rise to all $AdS_4$ massless representations. Here we extend this result to any dimension.

Let us start with the analysis of the scalar case of $|Rac\rangle$. The tensor product can be obtained by virtue of doubling of oscillators $Y_i^A \to Y_{i\,1}^A, Y_{i\,2}^A$ with the vacuum $|0\rangle$ satisfying

$$a_{1,2}^a|0\rangle = 0\,, \qquad \alpha_{1,2}|0\rangle = 0\,, \qquad \beta_{1,2}|0\rangle = 0\,. \tag{8.1}$$

The tensor product of the positive energy modules $|Rac\rangle^+ \otimes |Rac\rangle^+$ is spanned by the states of the form

$$|\Psi^+\rangle = \sum_{p,q,m,n} \frac{1}{2^{2(p+q)}q!p!(p+n+\frac{1}{2}M-1)!(q+m+\frac{1}{2}M-1)!} \times$$
$$\times (\bar{a}_{1l}\bar{a}_1{}^l)^p(\bar{a}_{2k}\bar{a}_2{}^k)^q \bar{\alpha}_1{}^{p+n+\frac{1}{2}M-1}\bar{\beta}_1{}^p\bar{\alpha}_2{}^{q+m+\frac{1}{2}M-1}\bar{\beta}_2{}^q\psi_{nm}^+(\bar{a}_1,\bar{a}_2)|0\rangle\,, \tag{8.2}$$

where

$$\psi_{pq}^+(\bar{a}_1,\bar{a}_2) = \psi^\pm_{m_1\ldots m_p\,n_1\ldots n_q}\bar{a}_1^{m_1}\ldots\bar{a}_1^{m_p}\bar{a}_2^{n_1}\ldots\bar{a}_2^{n_q}\,,$$
$$\psi^\pm_{m_1\ldots m_p\,n_1\ldots n_q}\eta^{m_1m_2} = 0\,, \qquad \psi^\pm_{m_1\ldots m_p\,n_1\ldots n_q}\eta^{n_1n_2} = 0\,. \tag{8.3}$$

$|\Psi^+\rangle$ satisfies

$$t_{1\,ij}|\Psi^+\rangle = t_{2\,ij}|\Psi^+\rangle = 0\,. \tag{8.4}$$

$|Rac\rangle^+ \otimes |Rac\rangle^+$ forms a bounded energy unitary module of the HS algebra $hu(1|(1,2){:}[M,2])$.

By virtue of (6.40) one observes that the lowest energy states annihilated by $T^-$ are $|\Psi^+(\psi(\bar{a}_1,\bar{a}_2)\rangle$ with $\psi(\bar{a}_1,\bar{a}_2)$ satisfying

$$\left(\frac{\partial}{\partial \bar{a}_1^a} + \frac{\partial}{\partial \bar{a}_2^a}\right)\psi(\bar{a}_1,\bar{a}_2) = 0\,, \tag{8.5}$$

i.e., those with

$$\psi(\bar{a}_1,\bar{a}_2) = \psi_0(\bar{a}_1 - \bar{a}_2)\,, \tag{8.6}$$

where $\psi_0(\bar{a})$ is an arbitrary harmonic polynomial. Using (6.41) one finds that the lowest energies are

$$E_0 = s + M - 2\,, \tag{8.7}$$

where $s$ is a degree of the polynomial $\psi_0(\bar{a})$. Therefore

$$|Rac\rangle \otimes |Rac\rangle = \sum_{s=0}^{\infty} \oplus \mathcal{H}(s+M-2, s, 0, 0\ldots)\,. \tag{8.8}$$

According to (1.7), the right hand side of this formula describes for $M>2$ the direct sum of all totally symmetric massless spin $s$ representations of the $AdS_{M+1}$ algebra $o(M,2)$. As a result, the tensor product of the massless scalar representation of the conformal group in $d-1$ dimensions with $d>3$ is shown to contain all integer spin totally symmetric massless states in $AdS_d$, that extends the result of Flato and Fronsdal [1] to any dimension. This spectrum of spins exactly corresponds to that of the model of [2] that proves that the HS symmetry $hu(1|2{:}[M,2])$ of [2] admits a unitary representation with the necessary spin spectrum, thus satisfying the admissibility condition.

The following comments are now in order.

The spin zero field in the $AdS_d$ HS multiplet has energy $d-3$ which is different from the energy of conformal scalar $\frac{1}{2}d-1$ beyond the case of $d=4$. This is not occasional because totally symmetric massless fields are not conformal for $d\neq 4$. It is therefore debatable whether or not one should call this scalar field massless. We will call it symmetrically massless scalar to emphasize that it belongs to the HS multiplet of symmetric massless fields and can be thought of as described by a degenerate zero-length one row Young tableau. Other "massless" scalar fields with energies $d-2-p$ can be thought of as degenerate cases of mixed symmetry gauge fields associated with $o(d-1)$-modules described by Young tableaux with $p<\frac{1}{2}d$ cells in the shortest column. Conformal fields are those with the highest possible $p=\frac{1}{2}d-1$ ($d$ is even).

The case of $M=2$ (i.e., bulk $AdS_3$) is special[9] because the right hand side of (8.8) contains representations corresponding to $3d$ singletons, i.e. $2d$ massless fields of all integer spins. Indeed, in accordance with the discussion of subsection 1.1, the lowest energy modules of $o(2,2)$ with the vacuum space being a $o(2)$-module described by Young tableaux of

[9] I am grateful to Aleksandr Gorsky for stimulating discussion of $AdS_3$ singletons.

height 1 are $2d$ conformal fields. Thus, the bilinear tensor product of $2d$ conformal scalars gives all integer spin $2d$ conformal fields. Note that this fact fits the admissibility condition because the $3d$ HS gauge field dynamics is of Chern-Simons type [55] so that HS gauge fields describe no bulk degrees of freedom analogously to the case of $3d$ gravity [63]. The obtained group-theoretical result indicates however that topological $3d$ HS interactions should have some dynamically nontrivial boundary manifestation in terms of $2d$ massless fields of all spins. It would interesting to work out a dynamical realization of this phenomenon.

Let the singleton module be endowed with a Chan-Paton index $|Rac\rangle \to |Rac\rangle^u$, $u = 1 \ldots n$. One can single out the symmetric and antisymmetric parts $(|Rac\rangle^u \otimes |Rac\rangle^v)_S$ and $(|Rac\rangle^u \otimes |Rac\rangle^v)_A$ of the tensor product $|Rac\rangle^u \otimes |Rac\rangle^v$. Since a permutation of the tensor factors exchanges both the oscillators $\bar{a}_1$ and $\bar{a}_2$ and the Chan-Paton indices, it follows that even (odd) spins in $(|Rac\rangle^u \otimes |Rac\rangle^v)_S$ and $(|Rac\rangle^u \otimes |Rac\rangle^v)_A$ are, respectively, symmetric (antisymmetric) and antisymmetric (symmetric) in the Chan-Paton indices. This pattern exactly corresponds to that of the HS gauge theories of [2] based on the HS gauge algebras $ho(n, 0|2{:}[M, 2])$ and $husp(n, 0|2{:}[M, 2])$, respectively. The massless states in the unsymmetrized tensor product $|Rac\rangle^u \otimes |Rac\rangle^v$ correspond to the HS gauge theory based on the HS gauge algebra $hu(n, 0|2{:}[M, 2])$.

Analogously one can consider higher rank tensor products of $|Rac\rangle$. In the rank $k$ tensor product, the lowest energy states are described by various polynomials $\psi^+(\bar{a}_1, \ldots, \bar{a}_k)$ which are "translationally invariant" $\sum_i \frac{\partial}{\partial \bar{a}_i} \psi^+(\bar{a}_1, \ldots, \bar{a}_k) = 0$ and harmonic with respect to each variable $\bar{a}_i$. For a degree $p$ polynomial in a rank $k$ tensor product, the $AdS_{d+1}$ lowest energy is $E_0 = p + \frac{1}{2}k(d-2)$. Comparing this formula with the lowest energies for massless fields in $AdS_{d+1}$ one finds that all states in the rank $k > 2$ tensor products of $|Rac\rangle$ are massive.

To analyze $|Di\rangle_\pm \otimes |Rac\rangle$ one observes by virtue of (6.40), (6.41), (7.28) and (7.29) that the lowest energy states in $|Di\rangle_\pm \otimes |Rac\rangle$ are described by $\gamma$-transverse (and, therefore, harmonic) tensor-spinors $\psi(\bar{a}_1 - \bar{a}_2)_{\pm\alpha}$. As a result,

$$|Di\rangle_\pm \otimes |Rac\rangle = \sum_{s=1/2,3/2\ldots} \oplus \mathcal{H}\left(s + M - 2, s, \frac{1}{2}, \frac{1}{2}, \ldots\right)_\pm , \tag{8.9}$$

where $n = s - \frac{1}{2}$ is a homogeneity degree of $\psi(\bar{a}_1 - \bar{a}_2)_{\pm\alpha}$. The right hand side of (8.9) contains all totally symmetric half-integer spin massless representations of $AdS_{M+1}$. This extends the corresponding $AdS_4$ result of Flato and Fronsdal to any dimension $d > 3$. In the special case of $M = 2$ the right hand side of (8.9) contains all $2d$ conformal fields of half-integer spins.

Let us now analyze a pattern of the tensor product of two conformal spinor representations. It is convenient to analyze $|Di\rangle_\rho \otimes {}_\kappa\langle Di|$ where $\rho$ and $\kappa$ are the chirality factors

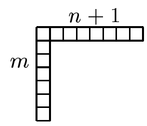
$${}_\kappa\langle \Psi^\pm|\Gamma = \kappa_\kappa\langle \Psi^\pm| , \qquad \Gamma|\Psi^\pm\rangle_\rho = \rho|\Psi^\pm\rangle_\rho , \qquad \rho^2 = \kappa^2 = 1 . \tag{8.10}$$

By virtue of (7.28) and (7.29) one finds that the lowest energy states in $|Di\rangle_\rho \otimes {}_\kappa\langle Di|$ are described by harmonic tensor bispinors $\psi(\bar{a}_1 + \bar{a}_2)_{\rho\kappa\alpha}{}^\beta$, which are $\gamma$-transverse both in $\alpha$

and in $\beta$. Equivalently, one can write

$$\psi(\bar{a})_{\rho\kappa\alpha}{}^{\beta} = \sum_{m=0}^{M}\sum_{n=0}^{\infty} \psi_{\rho\kappa}^{[b_1\dots b_m]|\{a_1\dots a_n\}} \gamma_{[b_1\dots b_m]\alpha}{}^{\beta}\, \bar{a}_{a_1}\dots\bar{a}_{a_n}\,, \tag{8.11}$$

where $\gamma_{[b_1\dots b_m]}$ are totally antisymmetrized products of $\gamma$ matrices. $\psi_{\rho\kappa}^{[b_1\dots b_m]\{a_1\dots a_n\}}$ is totally symmetric in the indices $a$ and totally antisymmetric in the indices $b$. It is easy to see that the left and right $\gamma$-transversality imply that $\psi_{\rho\kappa}^{[b_1\dots b_m]\{a_1\dots a_n\}}$ is traceless and that antisymmetrization over any $m+1$ indices must give zero. In other words it is described by the following traceless Young tableau of $o(M)$




. (8.12)

According to (7.29) the energies of these states are

$$\begin{aligned} E_0 &= n+M-1 = s+M-2\,, && s>0,\quad m<M \\ E_0 &= M-1\,, && s=0\,,\quad \text{or}\quad s=1,\quad m=M\,, \end{aligned} \tag{8.13}$$

where $s$ is the length of the upper row of the tableau (8.12). This means that all states with $s>0$ in the tensor product are massless except for those with $s=1, M>m>0$ and $s=0$

$$\begin{aligned} |Di\rangle\otimes\langle Di| = 2\mathcal{H}(M-1,0,0\dots)\oplus\sum_{s=1}^{\infty}\oplus\Bigg(2\sum_{m=0}^{[\frac{M}{2}]}\mathcal{H}(s+M-2,s,\underbrace{1,1\dots1}_{m},0,0\dots)\oplus \\ \oplus\,\mathcal{H}(s+M-2,s,\underbrace{1,1\dots1}_{M/2-1})\oplus \\ \oplus\,\mathcal{H}(s+M-2,s,\underbrace{1,1\dots1}_{M/2-2},-1)\Bigg)\,, \end{aligned} \tag{8.14}$$

where the last two terms appear only for even $M$ when the (anti)selfduality condition can be imposed. The appearance of massive totally antisymmetric fields $\mathcal{H}(M-1,1,1,\dots,0,0\dots)$ (which, however, become massless in the flat limit) in the higher dimensional HS multiplets is analogous to the case of the spin 0 field in $AdS_4$ [1].

The formula (8.14) is true for Dirac vacuum spinors. Imposing the chirality conditions for even $M$ we have the type A situation with the opposite chiralities and type B case with the same chiralities. In the type A case the column in (8.12) contains an odd number of cells while in the type B situation the column in (8.12) contains an even numbers of cells. In addition, the tableaux (8.12) with $m$ cells in a column are equivalent (dual) to those with $M-m$ cells. In particular, in the type B case the representation with $m=M/2$ is selfdual or antiselfdual depending on the chirality $\rho$ of $|Di\rangle_\rho$. For the special case of $M=2$ we obtain all totally symmetric integer spin $s>0$ $2d$ conformal fields in the type $A$

case. For the Dirac or type $B$ $M=2$ case, the tensor product of two $2d$ conformal spinor modules contains a $3d$ massive scalar field with $E_0=1$. It would be interesting to see what is a field-theoretical realization of this system.

For odd $M$ the operator $\Gamma$ is central. As a result, only the type B case is nontrivial for odd $M$. Since the corresponding tableaux (8.12) are selfdual, the resulting expansion contains all inequivalent representations in a single copy.

Using the results of [41], we conclude that the list of gauge fields resulting from gauging of $hu(1,1|(0,1,2){:}[M,2])$ just matches the list of massless states in the tensor product $(|Rac\rangle \oplus |Di\rangle) \otimes (\langle Rac| \oplus \langle Di|)$. Thus, the superalgebra $hu(1,1|(0,1,2){:}[M,2])$ and its chiral versions satisfy the admissibility condition and therefore are expected to give rise to consistent supersymmetric HS gauge theories in any dimension with totally symmetric fermionic massless fields of all half-integer spins. Leaving details of the exact formulation for a future publication let us mention that the form of the nonlinear dynamical equations for the supersymmetric case is essentially the same as that of [2] for the purely bosonic case modulo extension of the generating elements of the algebra with the Clifford fermions $\phi^A$ and spinor generating elements $\chi_\mu$ and $\bar{\chi}^\mu$. The same is true for the algebras $hu(n,m|(0,1,2){:}[M,2])$ with the nontrivial (spin 1) Yang-Mills algebra $u(n)\oplus u(m)$ and their orthogonal and symplectic reductions.

It is worth to note that although HS theories based on the superalgebra $hu(1,1|(0,1,2){:}[M,2])$ are supersymmetric in the HS sense, they are not necessarily supersymmetric in the standard sense. As explained in section 5, $hu(1,1|(0,1,2){:}[M,2])$ contains usual AdS superalgebras as subalgebras only for some lower $M$.

## 9. Unfolded equations for conformal fields

As pointed out in [8], there is a duality between unitary modules of single-particle quantum states and nonunitary modules underlying classical field equations. In [14, 16] it was shown for the $3d$ and $4d$ conformal systems that the corresponding duality has a form of certain (nonunitary) Bogolyubov transform. Let us show that the same is true in any dimension by deriving unfolded form of free conformal massless equations in $M$ dimensions.

Let us introduce the following basis of oscillators: $y^\pm = Y_1^M \pm Y_1^{M+1}$, $p^\pm = \frac{i}{2}(Y_2^M \pm Y_2^{M+1})$ and $y^n = Y_1^n$, $p^n = \frac{i}{2}Y_2^n$ with $n=0,\dots,M-1$ being $M$-dimensional Lorentz indices. The nonzero commutation relations are

$$[y^n,p^m] = -\eta^{nm}\,, \qquad [y^\pm,p^\pm]=0\,, \qquad [y^\pm,p^\mp]=2\,, \tag{9.1}$$

where $\eta^{nm}$ is the Minkowski metric with the signature $(1,M-1)$. Now we introduce a non-unitary Fock module $F$ with the vacuum state

$$p^n|0\rangle = 0\,, \qquad y^\pm|0\rangle = 0\,. \tag{9.2}$$

Its general element is

$$|\Phi\rangle = \phi(y^n,p^+,p^-)|0\rangle\,. \tag{9.3}$$

The submodule $SF$ to be associated with the classical scalar field in $M$ dimensions is spanned by the $sp(2)$ invariant states satisfying $t_{ij}|\Phi\rangle = 0$. To describe a massless scalar field in the Minkowski space $R^M$ we consider sections of the trivial fiber bundle $R^M \times SF$

$$|\Phi(x)\rangle = \phi(y^n, p^+, p^-|x)|0\rangle \,. \tag{9.4}$$

The conditions $t_{ij}|\Phi\rangle = 0$ imply that $\phi(y^n, p^+, p^-|x)$ has a form analogous to (6.14) and (6.15)

$$\phi(y, p^\pm|x) = \phi^+(y, p^\pm|x) + \phi^-(y, p^\pm|x) \tag{9.5}$$

with

$$\phi^+(y, p^\pm|x) = \sum_{p,n} \frac{1}{2^{2p}p!(p+n+\frac{1}{2}M-1)!}(y^m y_m)^p (p^+)^{p+n+\frac{1}{2}M-1}(p^-)^p \phi_n^+(y|x)\,, \tag{9.6}$$

$$\phi^-(y, p^\pm|x) = \sum_{p,n} \frac{1}{2^{2p}p!(p+n+\frac{1}{2}M-1)!}(y^m y_m)^p (p^-)^{p+n+\frac{1}{2}M-1}(p^+)^p \phi_n^+(y|x) \tag{9.7}$$

and $\phi_n^\pm(y|x)$ are arbitrary degree $n$ harmonic polynomials of $y$, i.e.

$$\phi_n^\pm(y|x) = \phi^\pm_{m_1\dots m_n} y^{m_1} \dots y^{m_n}\,, \qquad \psi^\pm_{klm_3\dots m_n}\eta^{kl} = 0\,. \tag{9.8}$$

To unfold some dynamical equations means to reformulate them in the form of appropriate zero curvature and covariant constancy conditions (for more details we refer the reader to the original paper [64] and to [65, 66]). In particular, the equations for matter fields and massless fields reformulated in terms of covariant curvatures (like, for example, Maxwell equations in terms of field strengths) have a form of covariant constancy equations on certain 0-forms called Weyl 0-forms. This name is borrowed from gravity where the corresponding covariant constancy equations describe differential restrictions on the Weyl tensor and all its derivatives. The Weyl 0-forms are sections of the fiber bundle over space-time with the fiber space dual to the space of single-particle quantum states by a Bogolyubov transform.

In our case the unfolded equations are

$$D|\Phi^+(x)\rangle = 0\,, \tag{9.9}$$

where $D = d + \omega_0$ is the covariant derivative with a flat connection $\omega_0$,

$$D^2 = 0\,, \tag{9.10}$$

which takes values in the conformal algebra $o(M,2)$ acting on the fiber module $SF$. To describe conformal field equations in flat (i.e., Minkowski) space-time one chooses $\omega_0$ to take values in the Poincaré subalgebra of the conformal algebra. To use cartesian coordinates, one takes the connection $\omega_0$ in the form $\omega_0 = dx^n P_n$, where $P_n$ are generators of translations of the Poincaré algebra. In our case, $P_n = y^- p_n$ and the equation (9.9) gets the form

$$dx^n \left(\frac{\partial}{\partial x^n} + y^- p_n\right)|\Phi^+(x)\rangle = 0\,. \tag{9.11}$$

In terms of components $\phi_n^+(y|x)$ it is equivalent to the infinite chain of equations

$$dx^m \left( \frac{\partial}{\partial x^m} \phi_n^+(y|x) + \frac{\partial}{\partial y^m} \phi_{n+1}^+(y|x) \right) = 0 \tag{9.12}$$

which expresses all higher components $\phi_n^+(y|x)$ with $n > 0$ via higher $x$-derivatives of $\phi_0^+(x)$ identified with the physical scalar field which satisfies the Klein-Gordon equation as a result of the conditions (9.8). The case of a scalar field in any dimension was considered in detail in [67]. Let us note that from the unfolded form of the massless scalar field equation interpreted as a covariant constancy condition it immediately follows (see e.g. [66]) that the massless scalar field equation is invariant under the global symmetry algebra $hu(1|2{:}[M,2])$ that provides an elementary proof of the result obtained by Eastwood [32].

Unfolded form of the fermionic massless equations is obtained analogously by using the spinorial module $|\Phi\rangle_\nu$ and the realization (2.15) of the conformal generators. The resulting equations can be found in [68] where the unfolded reformulation of all possible conformal field equations was given.

## 10. Conclusion

It is shown that $AdS_d$ HS global symmetry algebras underlying HS gauge theories of totally symmetric massless fields in $AdS_d$ of [2] admit unitary representations with the spectra of states matching those of the respective field-theoretical HS models. The states of the $AdS_d$ HS models of [2] correspond to the tensor product of the singleton modules identified with the space of single-particle states of the conformal scalar field in $d-1$ dimension. This fact extends the original observation of Flato and Fronsdal for $AdS_4$ [1] to any dimension and provides a group-theoretical basis for the $AdS/CFT$ correspondence between conformal boundary models and $AdS_d$ bulk HS theories. The group-theoretical analysis of this paper fits the field-theoretical analysis of the $AdS/CFT$ correspondence between bulk HS models and boundary conformal models of scalar fields carried out in [23, 26], based on the observation that conserved currents built of a massless scalar in $d$ dimensions match the list of on-mass-shell HS gauge fields in the $d+1$ dimensional bulk. In particular, the bilocal field introduced in [23] is a field-theoretical realization of the tensor product of a pair of singletons.

The extension to the supersymmetric case gives rise to HS superalgebras acting on the boundary conformal scalar and spinor as well as on infinite sets of totally symmetric massless bulk bosons and fermions and mixed symmetry massless fields described by hook tableaux with one row and one column. We argue that the bulk HS theories associated with the conformal spinor fields on the boundary are described by the nonlinear field equations having essentially the same form as that of [2] for totally symmetric massless fields. An interesting project for the future is to investigate whether there exists a generalization of the obtained results to a broader class of HS gauge fields in the bulk, which correspond to massless representations of a generic mixed symmetry type in $AdS_d$. From the field-theoretical side this requires further study of the mixed symmetry gauge fields in any dimension because, despite considerable progress achieved in the literature [69, 41], the



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full covariant formulation in $AdS_d$ is still lacking even at the free field level for generic $d$. Note that the full formulation of free totally symmetric massless fields in $AdS_d$ was obtained in [70, 60] in terms of HS gauge connections and in [71, 72] by using the BRST formalism. Also it is worth to mention that the structures of generic massless mixed symmetry fields in flat space and $AdS_d$ are essentially different: an irreducible massless field in $AdS_d$ decomposes into a family of massless fields in the flat limit [73].

Finally, let us mention that the case of $AdS_3$ singletons is special because bilinear tensor products of $2d$ conformal scalar and spinor contain infinite sets of $d = 2$ HS fields rather than $d + 1$ fields as it happens for all $d > 2$. This fact agrees with the field-theoretical description because HS gauge field dynamics in $AdS_3$ is of Chern-Simons type [55] so that HS gauge fields describe no bulk degrees of freedom. The obtained group-theoretical result indicates however that topological $3d$ HS interactions should have some dynamically nontrivial boundary manifestation in terms of $2d$ massless fields of all spins.


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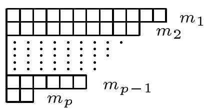
## A. Young tableaux

For the reader's convenience we summarize here some elementary properties of Young tableaux.

Let a $sl_M$ tensor $A^{a^1_1 \dots a^1_{m_1}, a^2_1 \dots a^2_{m_2}, a^3_1 \dots a^3_{m_3} \dots a^p_1 \dots a^p_{m_p}}$ be symmetric in the indices $a^i_k$ for any fixed $i$. It corresponds to the representation of $sl_M$ described by the Young tableau $Y(m_1, m_2, \dots m_p)$




(A.1)

if the tensor $A$ is such that, for any fixed $i$, total symmetrization of all indices $a^i_k$ with any index $a^l_j$ such that $l > i$ gives zero. Let $y^i_a$ be auxiliary variables with $a = 1, 2 \dots M$, $i = 1, 2 \dots p$. The tensors $A^{a^1_1 \dots a^1_{m_1}, a^2_1 \dots a^2_{m_2}, \dots a^p_1 \dots a^p_{m_p}}$ can be identified with the coefficients of the polynomials

$$A(y) = \sum_{m_1, m_2, \dots}^{\infty} A^{a^1_1 \dots a^1_{m_1}, a^2_1 \dots a^2_{m_2}, \dots a^p_1 \dots a^p_{m_p}} y^1_{a^1_1} \dots y^1_{a^1_{m_1}} y^2_{a^2_1} \dots y^2_{a^2_{m_2}} \dots y^p_{a^p_1} \dots y^p_{a^p_{m_p}} . \tag{A.2}$$

Note that the polynomials $A(y)$ can be written as

$$A(y) = \sum_{n=0}^{\infty} A^{a_1 \dots a_n}_{i_1 \dots i_n} y^{i_1}_{a_1} \dots y^{i_n}_{a_n} . \tag{A.3}$$

In these terms, the Young conditions take the simple form

$$y^i_a \frac{\partial}{\partial y^j_a} A(y) = 0 \qquad i < j\,, \tag{A.4}$$

$$y^i_a \frac{\partial}{\partial y^i_a} A(y) = m_i P(y) \qquad \text{no summation over } i\,. \tag{A.5}$$

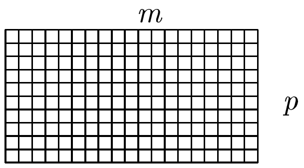
The operators

$$t_a{}^b = y^i_a \frac{\partial}{\partial y^i_b} \tag{A.6}$$

and

$$l^i{}_j = y^i_a \frac{\partial}{\partial y^j_a} \tag{A.7}$$

form the algebras $gl_M$ and $gl_p$, respectively. They are mutually commuting and are called Howe dual. It is useful to observe that the conditions (A.4) are the highest weight conditions for the algebra $sl_p \subset gl_p$. The $gl_M$ invariant conditions (A.5) fix some (integral) highest weight of $sl_p$ together with an eigenvalue of the central element of $gl_p$. Rectangular Young tableaux



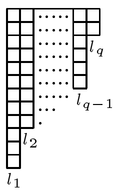

(A.8)

with $m_i = m$, which we call blocks, have a special property that they are $sl_p$ singlets

$$y^i_a \frac{\partial}{\partial y^j_a} P(y) = \frac{1}{p} \delta^i_j y^k_a \frac{\partial}{\partial y^k_a} P(y)\,. \tag{A.9}$$

(Note that (A.9) is a consequence of (A.4), (A.5) with $m_i = const$ along with the fact that the representations are finite dimensional because $A(y)$ is a polynomial. Combinatorial proof of this fact in terms of components of tensors is also elementary).

From the definition of the Young tableau $Y(m_1, m_2, \ldots m_p)$ it follows that $m_1 \geq m_2 \geq m_3 \ldots$ (otherwise the corresponding tensors are zero). For the same space we will also use notation with square brackets $Y[l_1, l_2, \ldots]$ where $l_1, l_2, \ldots$ are heights of columns




(A.10)

Obviously, one has $l_1 \geq l_2 \geq l_3 \ldots$ and $l_1 \leq M$ (because antisymmetrization over any $M+1$ indices $a$ taking $M$ values gives zero).

The realization of $Y[l_1, l_2, \ldots l_q]$ with manifest antisymmetrization is achieved in terms of polynomials $F(\phi^\alpha_a)$ of fermions

$$\phi^\alpha_a \phi^\beta_b = -\phi^\beta_b \phi^\alpha_a\,, \qquad \alpha, \beta = 1 \ldots q\,. \tag{A.11}$$



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The Young properties equivalent to (A.4) and (A.5) are

$$\phi_a^\alpha \frac{\partial}{\partial \phi_a^\beta} F(\phi) = 0 \qquad \alpha < \beta\,, \tag{A.12}$$

$$\phi_a^\alpha \frac{\partial}{\partial \phi_a^\alpha} F(y) = l_\alpha F(y) \qquad \text{no summation over } \alpha\,. \tag{A.13}$$

The coefficients $F^{a_1^1 \dots a_{l_1}^1, a_1^2 \dots a_{l_2}^2, \dots a_1^p \dots a_{l_q}^q}$ of

$$F(\phi) = \sum_{l_1, l_2, \dots, l_q}^{\infty} F^{a_1^1 \dots a_{l_1}^1, a_1^2 \dots a_{l_2}^2, \dots a_1^p \dots a_{l_q}^q} \phi^1_{a_1^1} \dots \phi^1_{a_{l_1}^1} \phi^2_{a_1^2} \dots \phi^2_{a_{l_2}^2} \dots \phi^q_{a_1^q} \dots \phi^q_{a_{l_q}^q} \tag{A.14}$$

are manifestly antisymmetric in the groups of indices $a_k^\alpha$ with fixed $\alpha$ associated with the $\alpha^{th}$ columns of the tableau. The condition (A.12) requires that total antisymmetrization of all indices associated with some column with any index from any subsequent column gives zero. Again, the conditions (A.12) and (A.13) are highest weight conditions for the algebra $gl_q$, formed by the operators

$$f^\alpha{}_\beta = \phi_a^\alpha \frac{\partial}{\partial \phi_a^\beta}\,, \tag{A.15}$$

which is Howe dual to the $gl_M$ formed by

$$s_a{}^b = \phi_a^\alpha \frac{\partial}{\partial \phi_b^\alpha}\,. \tag{A.16}$$

Rectangular (block) tableaux are $sl_q$ singlets.

It is not hard to see that the spaces of tensors with manifest symmetry and antisymmetry, $Y(m_1, m_2, \dots m_p)$ and $Y[l_1, l_2, \dots l_q]$, respectively, associated with the same Young tableau, are isomorphic.

If one is interested in representations of $o(M)$ rather than $sl_M$, the set of conditions (A.4), (A.5) is supplemented with

$$\eta_{ab} \frac{\partial^2}{\partial y_a^j \partial y_b^i} A(y) = 0\,, \tag{A.17}$$

where $\eta^{ab}$ is the $o(M)$ invariant metric. This is simply the condition that all $o(M)$ tensors in the decomposition (A.2) are traceless. For such spaces we use notation $Y^{tr}(m_1, m_2, \dots m_p)$. Note that for $o(M)$, the bosonic Howe dual algebra extends to $sp(2p)$ generated by (A.7) along with the operators

$$l^{ij} = \eta^{ab} y_a^i y_b^j\,, \qquad l_{ij} = \eta_{ab} \frac{\partial^2}{\partial y_a^j \partial y_b^i}\,, \tag{A.18}$$

which form the standard oscillator realization of $sp(2p)$. In the fermionic realization the equivalent condition is

$$\eta_{ab} \frac{\partial^2}{\partial \phi_a^\alpha \partial \phi_b^\beta} F(\phi) = 0\,. \tag{A.19}$$

The fermionic Howe dual algebra of $o(M)$ is $o(2q)$.

An important fact proved in section 2 is that

$$Y^{tr}[l_1, l_2, \dots l_q] = 0 \qquad \text{if } l_1 + l_2 > M. \tag{A.20}$$

Note that this identity is insensitive to the full Young properties. The only important properties are the tracelessness and total antisymmetry of each of the two groups of indices which together have more than $M$ indices.

Traceless $o(M)$ tableaux can be dualized by contracting the $\epsilon$ symbol with the first column. For even $M$, one can define (anti)selfdual tableaux $Y^{tr}_{\pm}[M/2, l_2, l_3, \dots, l_q]$ with the height of the first column $M/2$. Note that for any Young tableau there is at most one way to define selfduality because the maximal vertical block in the antisymmetric basis is symmetric with respect to its column interchange.

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